Dislocation Love numbers determined by satellite gravity missions

Wenke Sun and Shuhei Okubo
Earthquake Research Institute, University of Tokyo, Yayoi 1-1-1, Bunkyo-ku
Tokyo 113-0032, Japan, e-mail: sunw@eri.u-tokyo.ac.jp

Abstract:
This paper presents a new approach to calculate dislocation Love numbers using observations of a satellite gravity mission (e.g. GRACE). The necessary condition is that the co-seismic potential change be sufficiently large to be detected by the gravity mission. Co-seismic deformations for each spherical harmonic degree *n* are decoupled. Therefore, dislocation Love numbers of degree *n* can be determined independently. The determinable maximum harmonic degree *n* depends on the seismic size, source type, magnitude, and the accuracy of a satellite gravity mission. For an arbitrary seismic source, all four types of dislocation Love numbers can be determined using data from only one seismic event because all deformation components are involved together. Only the concerned dislocation Love numbers can be computed for any one of the four types of sources.

1. Introduction

Dedicated satellite missions, such as The Gravity Recovery and Climate Experiment (GRACE) [*NRC*, 1997], are now available for gravity field determination from space. They provide an extremely accurate, global, and high-resolution estimate of constant and time-variable components of the earth's gravity field every 30 days over a 5-year period [*Wahr et al.*, 1998]. It is anticipated that the gravity missions will yield extremely wide geophysical applications in geosciences, with measurement of temporal gravity variations caused by various geophysical processes including atmospheric mass redistribution, ocean circulation, polar ice melting or aggregation, visco-elastic response of the Earth's lithosphere to past and present loads, and others [*Chao et al.*, 2000; *Chao*, 2003]. In addition to these processes, earthquakes can produce large global gravity perturbations that are detectable through analysis of gravity mission data. *Gross and Chao* [2001] investigated gravity perturbations using normal mode technique based on *Chao and Gross* [1987]. Comparing the degree amplitude spectra of some earthquakes with expected GRACE sensitivity, they concluded that co-seismic effects of great earthquakes such as the 1960 Chilean or 1964 Alaska events can cause global gravitational field changes that are sufficiently large as to be detectable by GRACE. *Sun and Okubo* [2004a, b] approached this problem from two perspectives. They derived theoretical formulations of co-seismic geoid and gravity changes and their degree variances, expressed by dislocation Love numbers. These expressions are achieved using the dislocation theory of *Sun and Okubo* [1993], for a spherical earth as it is expressed in the form of spherical harmonics. They investigated co-seismic geoid and gravity changes by observing the distribution of their degree variances in comparison to the expected sensitivity of satellite gravity missions. Co-seismic deformations for large earthquakes are discussed with respect to their detectability. Results indicate that both the gravity and geoid changes are near two orders of magnitude larger than the precession of gravity missions in low harmonic degrees. Based on those results, they derived the minimum magnitudes of earthquakes detectable by GRACE. Their conclusion was that co-seismic deformations for an earthquake with a seismic magnitude greater than m=7.5 (for tensile sources) and m=9.0 (for shear sources) are expected to be detectable by GRACE. Note that the dislocation Love numbers used in *Sun and Okubo* [2004a, b] are obtained conventionally for a spherically symmetric earth model such as the 1066A [*Gilbert and Dziewonski*, 1975] or the preliminary reference earth model (PREM) [*Dziewonski and Anderson*, 1981]. However, dislocation Love numbers calculated based on an earth model are theoretically different from those of the earth. On the other hand, *Okubo et al.* [2002] showed that the co-seismic deformations vary if the earth parameters are adjusted. That fact implies that the accuracy of the dislocation Love numbers depends directly on the rightness of the adopted earth model. If possible, it is better to determine them by real observations because they carry real information regarding the earth's structure. Satellite gravity missions provide the possibility of determining the dislocation Love numbers.

Therefore, dislocation Love numbers are considered in this study as unknown variables. They are derived from observations of satellite gravity missions such as GRACE. Then theoretical formulations are presented for determining the dislocation Love numbers. They are applicable to study the inner structure of the earth as a new approach or condition because the dislocation Love numbers derived independently from space carry real information about the earth's mass distribution. Numerical applications should be useful for a seismic event that is sufficiently large to be detectable from space.

2. Co-seismic Potential Change and Dislocation Love Numbers

Assume that an inclined point dislocation is located on the polar axis in a compressible and self-gravitating spherical earth, and that the fault line is in the direction of $\varphi = 0$ (Greenwich meridian). According to the quasi-static dislocation theory, the co-seismic potential change at an observation point (a, θ, φ) can be expressed as [*Sun and Okubo*, 1993]

$$\psi^{ij}(a,\theta,\varphi) = \sum_{n,m} k_{nm}^{ij} Y_n^m(\theta,\varphi) \cdot \nu_i n_j \frac{g_0 UdS}{a^2} \quad , \tag{1}$$

where k_{nm}^{ij} (related to the gravitational potential change) are the dislocation Love numbers, function of the spherical harmonic degree, order, source depth, and source type. Components of the slip vector and its normal on the infinitesimal fault area *dS* are ν_i and n_j with total dislocation *U*. Gravity on the earth surface is g_0, *a* is the Earth's radius, and $Y_n^m(\theta,\varphi)$ is the spherical harmonic function of degree *n* and order *m*. The so-called dislocation factor, $g_0 UdS / a^2$, defines the earthquake magnitude and gives the unit of potential change.

A combination of the three slip and three normal components means that nine solutions exist for all possible sources. However, only four independent solutions exist if the earth model is spherically symmetric and isotropic. A deformation caused by an arbitrary source can be obtained by a proper combination of the four types of independent sources. In this study, we choose the following four independent solutions: *ij* = 12, 32, 22, and 33. They represent strike-slip, dip-slip, horizontal tensile and vertical tensile, respectively. Components of *ij* = 22 include two parts: *m* = 0 and 2. Computation of *m* = 2 is derived easily from the component of *ij* = 12. Expressions of the four independent solutions are given as the following [*Sun and Okubo,* 1993].

$$\psi^{12}(a,\theta,\varphi) = 2\sum_{n=2}^{\infty} k_{n2}^{12} P_n^2(\cos\theta)\sin 2\varphi \frac{g_0 UdS}{a^2} \tag{2}$$

$$\psi^{32}(a,\theta,\varphi) = 2\sum_{n=1}^{\infty} k_{n1}^{32} P_n^1(\cos\theta)\sin\varphi \frac{g_0 UdS}{a^2} \tag{3}$$

$$\psi^{22}(a,\theta,\varphi) = \left[\sum_{n=0}^{\infty} k_{n0}^{22} P_n(\cos\theta) - 2\sum_{n=2}^{\infty} k_{n2}^{12} P_n^2(\cos\theta)\cos 2\varphi\right] \frac{g_0 UdS}{a^2} \tag{4}$$

$$\psi^{33}(a,\theta,\varphi) = \sum_{n=0}^{\infty} k_{n0}^{33} P_n(\cos\theta) \frac{g_0 UdS}{a^2} \tag{5}$$

A co-seismic potential change caused by an arbitrary inclined fault can be expressed by the above four independent solutions. In this case, a dislocation vector $\mathbf{v}$ and its normal **n** can be described in terms of dip-angle δ and slip-angle λ of the fault as

$$\mathbf{n} = \mathbf{e}_3 \cos\delta - \mathbf{e}_2 \sin\delta \tag{6}$$

$$\mathbf{v} = \mathbf{e}_3 \sin\delta \sin\lambda + \mathbf{e}_1 \cos\lambda + \mathbf{e}_2 \cos\delta \sin\lambda \,. \tag{7}$$

We face a shear dislocation problem if dislocation vector $\mathbf{v}$ runs parallel to the fault plane. Similarly, for a tensile opening, the dislocation vector $\mathbf{v}$ and its normal **n** become equal:

$$\mathbf{v} = \mathbf{n} = \mathbf{e}_3 \cos\delta - \mathbf{e}_2 \sin\delta . \tag{8}$$

Then for an arbitrary shear fault on the polar axis, according to Eqs. (1)–(5) the co-seismic potential change can be written as the following.

$$\begin{aligned}\psi^{Shear}(a,\theta,\varphi) = \sum_{n=2}^{\infty} \Big\{ & (\sin\lambda \sin 2\delta \cos 2\varphi - 2\cos\lambda \sin\delta \sin 2\varphi) P_n^2(\cos\theta) k_{n2}^{12} \\ & + 2(\sin\lambda \cos 2\delta \sin\varphi + \cos\lambda \cos\delta \cos\varphi) P_n^1(\cos\theta) k_{n1}^{32} \\ & - \frac{1}{2}\sin\lambda \sin 2\delta P_n(\cos\theta) k_{n0}^{22} + \frac{1}{2}\sin\lambda \sin 2\delta P_n(\cos\theta) k_{n0}^{33} \Big\} \cdot \frac{g_0 U dS}{a^2}\end{aligned} \tag{9}$$

Similarly, for a tensile source, the co-seismic potential change becomes the following.

$$\begin{aligned}\psi^{Tensile}(a,\theta,\varphi) = \sum_{n=2}^{\infty} \Big[& \cos^2\delta P_n(\cos\theta) k_{n0}^{33} + \sin^2\delta P_n(\cos\theta) k_{n0}^{22} \\ & - 2\sin^2\delta \cos 2\varphi P_n^2(\cos\theta) k_{n2}^{12} - 2\sin 2\delta \sin\varphi P_n^1(\cos\theta) k_{n1}^{32} \Big] \cdot \frac{g_0 U dS}{a^2}\end{aligned} \tag{10}$$

Dislocation Love numbers k_{nm}^{ij} in (9) and (10) are obtained conventionally for a spherically symmetric earth model [*Sun and Okubo,* 1993] such as the 1066A [*Gilbert and Dziewonski*, 1975] or the preliminary reference earth model (PREM) [*Dziewonski and Anderson*, 1981]. Once a dislocation source or earthquake parameter is provided, co-seismic deformations can be calculated easily using these Love numbers. Subsequently, the potential change can be calculated by the above summations in (9) or (10).

However, the dislocation Love numbers calculated based on an earth model are theoretically worse than those of the earth. Furthermore, *Okubo et al.* [2002] showed that co-seismic deformations vary if the earth parameters are adjusted. That fact implies that the accuracy of the dislocation Love numbers depends directly on the adopted earth model. If possible, it is better to determine them using real observations. Satellite gravity missions provide just such a possibility. For that reason, dislocation Love numbers are considered to be unknown variables in this study, and are derived from observations of satellite gravity missions, such as GRACE. In previous studies, *Sun and Okubo* [2004a, b] proved that the satellite gravity mission (GRACE) is able to detect co-seismic deformations from space. This benefit allows the study of the earth's inner structure from a new vantage because dislocation Love numbers, when they are derived independently from space, carry reliable information of the earth's mass distribution. Therefore, the observed dislocation Love numbers are useful not only for geodetic application, but also for modeling the earth structure as a new condition, in combination with seismic knowledge.

Co-seismic deformations can be studied for individual harmonic degrees because the satellite gravity missions provide potential measurements in the form of spherical harmonic coefficients, as indicated by *Chao and Gross* [1987]. Note that the terms of degrees n=0 and n=1 in Eqs. (9) and (10) vanish because the total mass of the earth is constant and the reference frame origin is located at the center of mass of the earth model. On the other hand, the angular order m vanishes except m = 0, 1 and 2 because the source is chosen at the polar axis and because of the symmetric property of the source functions.

3. Potential Change Observed by Satellite Gravity Missions

On the other hand, satellite gravity missions provide the following observations for a potential anomaly as a spherical harmonic series [*Heiskanen and Moritz,* 1967]:

$$T(a,\theta',\varphi') = a \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (\Delta C_{nm} \cos m\varphi' + \Delta S_{nm} \sin m\varphi') P_n^m(\cos\theta') , \tag{11}$$

where ΔC_{nm} and ΔS_{nm} are differences of two sets of spherical harmonic coefficients (C^1_{nm} , S^1_{nm}) and (C^2_{nm} , S^2_{nm}) of the geo-potential model observed by the GRACE mission:

$$\Delta C_{nm} = C^2_{nm} - C^1_{nm}\text{, and} \tag{12}$$

$$\Delta S_{nm} = S^2_{nm} - S^1_{nm}\ . \tag{13}$$

Notice that a dislocation may occur at an arbitrary position in the earth in practice, whereas satellite gravity missions always provide results in spherical coordinates with the North Pole as orientation. Then the theoretical co-seismic potential changes expressed in (9) and (10) and the potential change provided by the satellite in (11) are for two different spherical coordinate systems: (a,θ,φ) and (a,θ',φ'). For comparison, they must be unified into one coordinate system by transforming one of them to the other's format. Because of the spherical symmetric property, results are identical whichever is transformed. Nevertheless, the dislocation Love numbers in (9) and (10) are unknown and are derived from satellite observations. For that reason, it is convenient to leave them unchanged. On the other hand, if the seismic source is chosen at a pole, the co-seismic potential change only contains spherical harmonic orders *m*=0, 1 and 2. Otherwise, all spherical harmonic orders *m* will be involved. Therefore, we transform Eq. (11) into the same system as that used in (9) and (10) below:

$$T(a,\theta,\varphi) = a\sum_{n=0}^{\infty}\sum_{k=-n}^{n}\left(\Delta c_{nk}\cos k\varphi + \Delta c_{nk}\sin k\varphi\right)P^k_n(\cos\theta)\,, \tag{14}$$

where

$$\Delta c_{nk} = \sum_{m=0}^{n} a^k_{nm}\Delta C_{nm} \tag{15}$$

$$\Delta s_{nk} = \sum_{m=0}^{n} b^k_{nm}\Delta S_{nm} \tag{16}$$

and coefficients a^k_{nm} and b^k_{nm} can be obtained by a set of reoccurrence formulas, as listed in Appendix A.

4. Dislocation Love Numbers k^{ij}_{nm} Derived From Observations of Gravity Missions

Theoretically, the predicted potential change $\psi^{Shear}(a,\theta,\varphi)$ (or $\psi^{Tensile}(a,\theta,\varphi)$) should be identical to the observed $T(a,\theta,\varphi)$ anywhere on the earth surface, as

$$\psi(a,\theta,\varphi) \equiv T(a,\theta,\varphi)\ . \tag{17}$$

In practice, the use of either $\psi^{Shear}(a,\theta,\varphi)$ or $\psi^{Tensile}(a,\theta,\varphi)$ depends on the actual source type – shear or tensile. On the other hand, relation (17) holds for any harmonic degree. Consequently, we only discuss spherical harmonic degree *n* in the following. Assuming a shear seismic source, inserting (9) and (14) into (17) yields

$$f_1(\theta,\varphi)k^{12}_{n2} + f_2(\theta,\varphi)k^{32}_{n1} + f_3(\theta,\varphi)k^{22}_{n0} + f_4(\theta,\varphi)k^{33}_{n0} = g(\theta,\varphi)\,, \tag{18}$$

where

$$f_1(\theta,\varphi) = \left(\sin\lambda \sin 2\delta \cos 2\varphi - 2\cos\lambda \sin\delta \sin 2\varphi\right) P_n^2(\cos\theta) \frac{g_0 UdS}{a^2} \tag{19}$$

$$f_2(\theta,\varphi) = 2\left(\sin\lambda \cos 2\delta \sin\varphi + \cos\lambda \cos\delta \cos\varphi\right) P_n^1(\cos\theta) \frac{g_0 UdS}{a^2} \tag{20}$$

$$f_3(\theta,\varphi) = -\frac{1}{2}\sin\lambda \sin 2\delta P_n(\cos\theta) \frac{g_0 UdS}{a^2} \tag{21}$$

$$f_4(\theta,\varphi) = \frac{1}{2}\sin\lambda \sin 2\delta P_n(\cos\theta) \frac{g_0 UdS}{a^2} \tag{22}$$

$$g(\theta,\varphi) = a \sum_{k=-n}^{n} \left(\Delta c_{nk} \cos k\varphi + \Delta c_{nk} \sin k\varphi\right) P_n^k(\cos\theta)\,. \tag{23}$$

Note that $f_i(\theta,\varphi)$ (i=1,2,3,4) and $g(\theta,\varphi)$ in (18) are known once the parameters of an earthquake are provided. The only unknowns are the four dislocation Love numbers k_{n2}^{12}, k_{n1}^{32}, k_{n0}^{22} and k_{n0}^{33}. The remaining task is to solve Eq. (18). For that purpose, (18) can be discretized into a linear system

$$\mathbf{FK} = \mathbf{G}\,, \tag{24}$$

where

$$\mathbf{K} = \left(k_{n2}^{12}, k_{n1}^{32}, k_{n0}^{22}, k_{n0}^{33}\right)^{\mathrm{T}} \tag{25}$$

$$\mathbf{G} = \left(g(\theta_1,\varphi_1), g(\theta_2,\varphi_2), \cdots g(\theta_N,\varphi_N)\right)^{\mathrm{T}} \tag{26}$$

$$\mathbf{F} = \begin{pmatrix} f_1(\theta_1,\varphi_1) & f_2(\theta_1,\varphi_1) & f_3(\theta_1,\varphi_1) & f_4(\theta_1,\varphi_1) \\ f_1(\theta_2,\varphi_2) & f_2(\theta_2,\varphi_2) & f_3(\theta_2,\varphi_2) & f_4(\theta_2,\varphi_2) \\ \cdots & \cdots & \cdots & \cdots \\ f_1(\theta_N,\varphi_N) & f_2(\theta_N,\varphi_N) & f_3(\theta_N,\varphi_N) & f_4(\theta_N,\varphi_N) \end{pmatrix} \tag{27}$$

So that the unknown dislocation Love numbers **K** can be obtained easily as

$$\mathbf{K} = \mathbf{F}^{-1}\mathbf{G}\,. \tag{28}$$

Decoupling of the dislocation Love numbers simplifies matters if the seismic source is occasionally one of the four independent types. For a vertical strike-slip (m=2), Eq. (18) can be reduced to

$$f_1(\theta,\varphi) k_{n2}^{12} = g(\theta,\varphi)\,. \tag{29}$$

Thereby, the dislocation Love number can be written as

$$k_{n2}^{12} = g(\theta,\varphi) / f_1(\theta,\varphi)\,. \tag{30}$$

Similarly, we may obtain dislocation Love numbers for the other three source types as

$$k_{n1}^{32} = g(\theta,\varphi) / f_2(\theta,\varphi) \tag{31}$$

$$k_{n0}^{22} = g(\theta,\varphi) / f_3(\theta,\varphi) \tag{32}$$

$$k_{n0}^{33} = g(\theta,\varphi) / f_4(\theta,\varphi)\,. \tag{33}$$

However, in this special case, not all, but only the concerned dislocation Love numbers can be determined.

For a tensile source, similar equations exist as (18)–(33), but with slightly different contents for $f_i(\theta,\varphi)$ (i=1,2,3,4). They are omitted here because a pure tensile rupture occurs very rarely in practice. If necessary, they can be written out easily in the same manner as those for the shear source.

5. Discussion and Final Remarks

This study presents a new method to calculate dislocation Love numbers using observations of a satellite gravity mission such as GRACE. A necessary condition is that the co-seismic deformations (e.g., potential change) should be sufficiently large to be detectable by the gravity mission. Deformations for each spherical harmonic degree *n* are decoupled. Therefore, the dislocation Love numbers can be determined independently for each *n*. However, the maximum determinable harmonic degree *n* depends on the seismic size, source type, and the accuracy (detectability) of a satellite gravity mission. For example, for a seismic event as large as the 1964 Alaska earthquake, the dislocation Love numbers of the first 70 spherical harmonic degrees are expected to be determinable using GRACE observations [*Sun and Okubo*, 2004a, b]. The forthcoming gravity mission GRACE follow-on is expected to improve accuracy by two orders [*Watkins, et al.*, 2000]. Therefore, more harmonic degrees are expected to be determined. On the other hand, for an arbitrary seismic source (with certain dip angle, e.g., around $\delta = 45^\circ$), all four types of dislocation Love numbers can be determined using only one seismic event because all deformation components are involved together. However, only the related dislocation Love numbers can be computed in the case of any one of the four types of sources. For example, only k_{n2}^{12} can be obtained for a vertical strike-slip earthquake.

Acknowledgments. This research was supported financially by a JSPS research grant (C16540377) and a grant for study of “Basic design and feasibility studies for the future missions for monitoring Earth’s environment”.

References

Chao, B. F., and R. S. Gross (1987), Changes in the Earth’s rotation and low-degree gravitational field induced by earthquakes, Geophys. J. R. Astr. Soc., *91*, 569–596.

Chao, B. F., V. Dehant, R. S. Gross, R. D. Ray, D. A. Salstein, M. M. Watkins, and C. R. Wilson (2000), Space geodesy monitors mass transports in global geophysical fluids, Eos, Transactions, American Geophysical Union, *81*, 247–250.

Chao, B. F. (2003), Geodesy is not just for static measurements any more, Eos, Transactions, American Geophysical Union, *84*, 145–156.

Dziewonski, A. M., and D. L. Anderson (1981), Preliminary Reference Earth Model, *Phys. Earth Planet. Inter.*, *25*, 297–356.

Gilbert, F., and A. M. Dziewonski (1975), An application of normal mode theory to the retrieval of structural parameters and source mechanisms from seismic spectra, *Phil. Trans. R. Soc. London A, 278*, 187–269.

Gross, R. S., and B. F. Chao (2001), The gravitational signature of earthquakes, in Gravity, Geoid, and Geodynamics 2000, edited by M.G. Sideris, pp. 205–210, IAG Symposia Vol. 123, Springer-Verlag, New York.

Heiskanen, W. H., and Z. Moritz (1967), Physical Geodesy, Freeman, San Francisco.

National Research Council, NAS, Satellite Gravity and the Geosphere, ed. J. O. Dickey, Washington, D.C., 1997.

Okubo, S., W. Sun, T. Yoshino, T. Kondo, J. Amagai, H. Kiuchi, Y. Koyama, R. Ichikawa, and M. Sekido (2002), Far-Field Deformation due to Volcanic Activity and Earthquake Swarm, Vistas for Geodesy in the New Millennium, Adam, J. and K.P. Schwarz (Eds.), International Association of Geodesy Symposia, Volume 125, 518-522.

Sun, W., and S. Okubo (1993), Surface potential and gravity changes due to internal dislocations in a spherical earth – I. Theory for a point dislocation, *Geophys. J. Int.*, *114*, 569–592.

Sun, W., and S. Okubo (2004a), Co-seismic Deformations Detectable by Satellite Gravity Missions – a Case Study of Alaska (1964, 2002) and Hokkaido (2003) Earthquakes in the Spectral Domain, J. Geophys. Res., Vol. 109, No. B4, B04405, doi:10.1029/2003JB002554.
Sun, W., and S. Okubo (2004b) Truncated Co-seismic Geoid and Gravity Changes in the Domain of Spherical Harmonic Degree, EPS, *56*, 881-892.
Wahr, J., M. Molenaar, and F. Bryan (1998), Time variability of the Earth's gravity field: Hydrological and oceanic effects and their possible detection using GRACE, J. Geophys. Res., *103*, 30205–30230.
Watkins, M. M., W. M. Folkner, B. F. Chao, and B. D. Tapley (2000), The NASA EX-5 Mission: A laser interferometer follow-on to GRACE, IAG Symp. GGG2000, Banff, July.
Xu, H., and F. Jiang, (1964), Transformation of spherical harmonic expressions of gravity anomaly, ACTA Geodetica et Cartographica Sinica, *7*, 252-260.

Appendix A. Transformation coefficients a_{nm}^k and b_{nm}^k

According to Xu and Jiang (1964), the transformation coefficients a_{nm}^k and b_{nm}^k used in (15) and (16) can be derived by the following recurrence formulas, assuming (θ_0, φ_0) as the orientation of the seismic source, i.e.

$$a_{nm}^0 = P_{nm}(\cos\theta_0)\,;\; b_{nm}^0 = 0$$

for k=1:

$$(n-m+1)a_{n+1,m}^1 = n\cos\theta_0 a_{nm}^1 + \sin\theta_0 a_{nm}^0 - \frac{1}{2}(n-1)n\sin\theta_0 a_{nm}^2$$

$$(n-m+1)b_{n+1,m}^1 = n\cos\theta_0 b_{nm}^1 + \sin\theta_0 b_{nm}^0 - \frac{1}{2}(n-1)n\sin\theta_0 b_{nm}^2$$

$$a_{n,n}^1 = (n-1)\sin\theta_0 a_{n-1,n-1}^1 + \left(b_{n-1,n-1}^0 - \cos\theta_0 a_{n-1,n-1}^0\right) + \frac{1}{2}(n-1)(n-2)\left(b_{n-1,n-1}^2 + \cos\theta_0 a_{n-1,n-1}^2\right)$$

$$b_{n,n}^1 = (n-1)\sin\theta_0 b_{n-1,n-1}^1 + \left(a_{n-1,n-1}^0 - \cos\theta_0 b_{n-1,n-1}^0\right) + \frac{1}{2}(n-1)(n-2)\left(a_{n-1,n-1}^2 + \cos\theta_0 b_{n-1,n-1}^2\right)$$

for k>=2:

$$(n-m+1)a_{n+1,m}^k = (n-k+1)\cos\theta_0 a_{nm}^k + \frac{1}{2}\sin\theta_0 a_{nm}^{k-1} - \frac{1}{2}(n-k)(n-k+1)\sin\theta_0 a_{nm}^{k+1}$$

$$(n-m+1)b_{n+1,m}^k = (n-k+1)\cos\theta_0 b_{nm}^k + \frac{1}{2}\sin\theta_0 b_{nm}^{k-1} - \frac{1}{2}(n-k)(n-k+1)\sin\theta_0 b_{nm}^{k+1}$$

$$a_{n,n}^k = (n-k)\sin\theta_0 a_{n-1,n-1}^k + \frac{1}{2}\left(b_{n-1,n-1}^{k-1} - \cos\theta_0 a_{n-1,n-1}^{k-1}\right) + \frac{1}{2}(n-k)(n-k-1)\left(b_{n-1,n-1}^{k+1} + \cos\theta_0 a_{n-1,n-1}^{k+1}\right)$$

$$b_{n,n}^k = (n-k)\sin\theta_0 b_{n-1,n-1}^k + \frac{1}{2}\left(a_{n-1,n-1}^{k-1} - \cos\theta_0 b_{n-1,n-1}^{k-1}\right) + \frac{1}{2}(n-k)(n-k-1)\left(a_{n-1,n-1}^{k+1} + \cos\theta_0 b_{n-1,n-1}^{k+1}\right)$$