ON NORMAL MODES CALCULATION OF A LATERALLY HETEROGENEOUS EARTH

- A NEW COMPUTATIONAL TECHNIQUE -

Tesis by

Hitoshi Kawakatsu

In Partial Fulfillment of the Requirements

for the Degree of

Master of Science

University of Tokyo

Tokyo , Japan

1981

(Submitted March 7, 1981)

Acknowledgements

I am grateful to Prof. Robert Geller. Without his suggestions and encouragements I could not finish this work on time. He also made me interested in seismology and Japan.

I am also grateful to my advisor, Prof. M. Saito. His suggestions were helpful to complete this work. He critically read the manuscript too.

The way of life of Prof. H. Takeuchi, my another advisor and his book titled "My exercises of intelligence" have taught me a lot.

Wordless encouragements by the last members of Takeuchi Lab. pushed me to finish the degree. Discussions with Drs. T. Matsui and K. Kurita will be a good seed of my next thesis. Mr. S. Okubo kindly taught me the usage of the computer terminal. Mr. K. Yomogida helped me to make the figures.

I was very fortunate to be able to stay at Stanford Univ. from September of 1979 to October of 1980 as an exchange student. All of the fuculty and student of Dept. of Geophysics had been extremely kind during my stay there. Great reputations of many famous Japanese geophysisists - such as, Profs. S. Uyeda, K. Aki, H. Kanamori, Y. Fukao etc. gave me a great deal of advantages as a prospective Japanese geophysisist.

Abstract

Ordinaly perturbation theories are not appropriate to deal with lateral heterogeneity of the earth because of selection rules. Although a variational method (Rayleigh-Ritz method) gives good approximation to normal modes of a laterally heterogeneous body, it is not realistic for the earth because we have to solve very large matrix eigenvalue problems.

A new computational technique is proposed. This technique is based on second-order perturbation theory and use it iteratively to compensate a loss of information about odd harmonic order heterogeneities in a model, which comes from selection rules of perturbation theory.

Vibration of a heterogeneous string and of a heterogeneous square membrane is examined with using this technique. Convergence of this technique is rapid and the results are remarkablly good. It is suggested that this technique may be applicable to the free oscillation of the heterogeneous earth.

Table of Contents

	Page
Acknowledgments	ii
Abstract	iii
1 Introduction	1
2 Eigenvalue problem	3
3 New technique	7
4 Numerical examples	
4.1 Heterogeneous String	9
4.2 Heterogeneous Membrane	16
5 Discussions	29
6 Conclusions	31
References	32

1 Introduction

How the lateral heterogeneity of the earth affects the normal modes of earth's free oscillation is a very interesting and important geophysical problem. However, because of computational limitations, it has not been solved completely.

Approaches which have been prososed are summerized as follows

(1) First-order degenerate perturbation theory by Madariaga (1972), Saito(1971) and Luh(1973) - which considers only the effect of coupling between modes within a single multiplet of unperturbed model.

(2) Quasi-degenerate perturbation theory by Luh(1973 and 1974) and Dahlen(1969) - which considers coupling of neiboring multiplets that have nearly equal unperturbed eigenfrequencies.

(3) Variational method by Usami(1971) and Plumlee & Geller (1980) - the eigenfunction being calculated is considered to be a linear combination of members of a complete set of functions.

(4) Finite element method by Stifler & Bolt(1981).

Geller & Stein(1978, hereafter referred to as G&S) compared three approaches (first-order and second-order perturbation theory and variational method) applying to vibration of a simple heterogeneous string, and concluded that Rayleigh-Ritz method was the most appropriate for the problem and also for the earth.

Plumlee & Geller(1980) extended the work doen by Geller

and Stein into a simple three dimentional case. They showed that "the strongest coupling is not necessarily within the modes which are neiboring in unperturbed frequency" and " in many cases, the coupling within modes of adjacent angular orders is much larger than coupling within modes of adjacent frequencies". This suggests that ordinary perturbation theories, such as (1) and (2), are not sufficient and the effect of coupling between multiplets should also be taken into account.

In the earth's case , however, a very serious computational problem arises if we apply Rayleigh-Ritz method proposed by G&S - i.e., we have to solve large eigenvalue problems.

The purpose of this thesis is to present a new computational technique, which makes things more realistic, with application to simple one and two dimensional problems.

2 Eigenvalue Problem

Because a brief (but enough) review of three approximation methods (the Rayleigh-Ritz method and first- and second-order perturbation theory) for a nondegenerate eigenvalue problem can be found in G&S (see appendix), it will not be repeated here and the same notation will be used. We have to be careful of some misprints in their paper. Sufixes of the denominators of equations (10) and (11) of G&S(i.e., "n" and "m") should be exchanged each other.

Whether the problem is the free oscillation of the earth or a vibration of a string, it can be reduced to the following eigenvalue problem,

$$H \Psi + \rho \omega^2 \Psi = 0 \qquad (1)$$

with boundary conditions

$$\alpha \Psi + \beta \frac{\partial \Psi}{\partial n} = 0$$

where H is an hermitian operator, ρ is the density, ω is the eigenfrequency, Ψ is the eigenfunction and α and β are arbitrary constants.

Heterogeneity is introduced into the operator and density as

$$H = H^{(0)} + H^{(1)}$$

and

$$\rho = \rho^{(0)} + \rho^{(1)}$$

where $H^{(0)}$ and $\rho^{(0)}$ are the unperturbed operator and density, and $H^{(1)}$ and $\rho^{(1)}$ are the corresponding perturbations. In the following the superscript 0 will be used to denote unperturbed quantities. The perturbed eigenfunction, Ψ_n, is expanded in terms of unperturbed eigenfunctions, $\Psi_m^{(0)}$, i.e.,

$$\Psi_n = \sum_m C_{nm} \Psi_m^{(0)}$$

where C_{nm} is the expansion coefficient. Then problem is reduced to find those expansion coefficients for each Ψ_n.

According to G&S, the second-order solutions is given by

$$C_{nm} = \frac{H_{mn}^{(1)} + (\omega_n^{(0)})^2 \rho_{mn}^{(1)}}{(\omega_m^{(0)})^2 - (\omega_n^{(0)})^2} \quad , \; m \neq n$$

$$C_{nn} = 1 \tag{2}$$

where

$$H_{ij}^{(1)} = \int (\Psi_i^{(0)})^* \, H^{(1)} \, (\Psi_j^{(0)}) \, dV$$

$$\rho_{ij}^{(1)} = \int (\Psi_i^{(0)})^* \, \rho^{(1)} \, (\Psi_j^{(0)}) \, dV \tag{3}$$

and variational method leads to an eigenvalue problem

$$V\,\mathbb{C} = \omega^2\, T\,\mathbb{C} \tag{4}$$

where $\mathbb{C}$ is the expansion coefficient and

$$\begin{aligned} T_{ij} &= \rho_{ij}^{(1)} + \delta_{ij} \\ V_{ij} &= -H_{ij}^{(1)} + \delta_{ij}\,(\omega_j^{(0)})^2 \end{aligned} \tag{5}$$

It should be noted that in both cases what define the problems are the coupling matrices, $H^{(1)}$, $\rho^{(1)}$, and the unperturbed eigenfrequencies, $\omega_n^{(0)}$.

In their paper, G&S applied these three methods to vibration of a simple heterogeneous string which has a discontinuity in density and stiffness at the center and is uniform elsewhere. Comparing with the exact solutions, they concluded that the variational method (Rayleigh-Ritz method) may be the most appropriate method for finding the normal modes of a laterally heterogeneous earth model.

Although Rayleigh-Ritz method gives a good approximation to normal modes of a laterally heterogeneous body as they showed, a serious computational problem appears when we apply the method to an earth model. Accuracy of the method depends on number of modes used for expansion. If we use an adequate number of modes for an earth model, the matrix eigenvalue problem, (4), will be very large. G&S suggested that inverse iteration method was useful to this kind of problem. In that

case we have to solve a large system of simultaneous equations at each iteration.

In this study a new technique is proposed to avoid these difficulties in Rayleigh-Ritz method.

3 New technique

In G&S, they apparently showed that even second-order perturbation theory was insufficient for eigenvalue problems of a laterally heterogeneous body - a variational approach was needed. However it may not be true. In Fig.1 of G&S, second-order theory is oscillating around the correct answer, even though it is off. And in Fig.2 of G&S, the expansion coefficients by second-order theory corresponding to the modes an odd distance away from the "parent" mode are just about the same as the exact solution for N = 1, 11 and 21, but not for N = 31, 41 and 51. The modes an even distance away all have zero coefficients because of selection rules. For N = 31, 41 and 51 the second-order eigenfunctions are poor because the original mode of unperturbed solution does not contribute much to the final answer.

The second-order theory does not give the right answer mainly because the selection rules exclude even modes. The fact that the second-order eigenfrequencies oscillate about the correct answer and that in the two methods (second-order perturbation theory and Rayleigh-Ritz method) what define the problems are same coupling matrices $H^{(1)}$, $\rho^{(1)}$, and unperturbed eigenfrequencies, $\omega_n^{(0)}$, means that several iterations of some kind of second-order theory may converge to the correct answer without solving large systems of simultaneous equations.

The following new iterative technique is proposed.

$$C_{nm}^{(k+1)} = \frac{\sum_i (H_{mi}^{(1)} + (\omega_n^{(k)})^2 \rho_{mi}^{(1)}) C_{ni}^{(k)}}{(\omega_m^{(0)})^2 - (\omega_n^{(k)})^2}, \quad m \neq n$$

$$C_{nn}^{(k+1)} = C_{nn}^{(k)}. \tag{6}$$

After each iteration the Rayleigh quotient is used to estimate corresponding eigenfrequencies - i.e.,

$$(\omega_n^{(k)})^2 = \frac{\mathbb{C}_n^t \, V \, \mathbb{C}_n}{\mathbb{C}_n^t \, T \, \mathbb{C}_n}$$

where $\mathbb{C}_n = (C_{n1}, \ldots\ldots, C_{nm}, \ldots\ldots)^t$ and $C_{nm}^{(0)} = \delta_{nm}$ (Kronecker's delta).

Because of the initial condition $C_{nm}^{(0)} = \delta_{nm}$, this technique gives the same results as the second-order theory at first iteration, but on further iterations third order correction develops, the forbidden modes no more forbidden. After convergence is obtained we normalize $\mathbb{C}_n$ by

$$R = \mathbb{C}_n^t \, T \, \mathbb{C}_n$$

and the normalized eigenfunction is given by

$$\Psi_n = \sum_m C_{nm} \, \Psi_m^{(0)} \, / \sqrt{R} \, .$$

4 Numerical examples

4.1 Heterogeneous String

In this section we consider the exactly same problem discussed in G&S, a one-dimensional string, with jump of stiffness, k, and density, ρ, at the center and uniform elastic properties elsewhere.

Equation of motion and boundary conditions are

$$\frac{d}{dx}\left(k(x)\,\frac{d\Psi(x)}{dx}\right) + \rho(x)\,\omega^2\,\Psi(x) = 0$$
$$\Psi(0) = \Psi(\ell) = 0 \qquad (7)$$

where

$$k(x) = k_0 - (\delta k)$$
$$\rho(x) = \rho_0 - (\delta\rho) \qquad 0 < x < \ell/2$$

$$k(x) = k_0 + (\delta k)$$
$$\rho(x) = \rho_0 + (\delta\rho) \qquad \ell/2 < x < \ell$$

Here k_0 and ρ_0 are the unperturbed stiffness and density, and δk and $\delta\rho$ are the perturbations and ℓ is the length of the string.

The coupling matrices are given by

$$\rho^{(1)}_{mn} = \frac{2(\delta\rho)}{\rho_0 \pi}\left[\frac{\sin\frac{(m+n)\pi}{2}}{m+n} - \frac{\sin\frac{(m-n)\pi}{2}}{m-n}\right] \qquad (8)$$

$$H_{mn}^{(1)} = \frac{2(\delta k)mn\pi}{\rho_o \ell^2} \left[\frac{\sin\frac{(m+n)\pi}{2}}{m+n} + \frac{\sin\frac{(m-n)\pi}{2}}{m-n} \right]$$

Because of selection rules $H_{mn}^{(1)}$ and $\rho_{mn}^{(1)}$ are equal to zero when (n-m) is even.

We used 41 trial functions as in G&S in the actual calculation. The exact solutions are calculated by the program used in G&S.

In Figure 1 and Figure 2, we considered first 21 modes on the exactly same problem as in G&S - i.e., k_0=1.0, ρ_0=0.5, $\delta k = \delta\rho$=0.1 and ℓ=20. Convergence of this technique is extremely good. Four iterations seem to be enough for modes up to N=21.

Since this technique is based on a perturbation theory, it may not give the right solution if the heterogeneity is too large. In Figure 3, we extended calculation for modes up to N=50 and changed the perturbation of elastic properties in order to see this effect. As expected, it gave wrong answer after N=40 for the same problem.(Figure 3-a) The plots of exact answer in Fig.2 of G&S shows that when N=31 or 41 the expansion coefficient for its own mode (i.e., N^{th} mode) is smaller than the others. Since such a phenomina may not occur in the earth case, this technique may be applicable to the earth. Figure 3-b and 3-c are the case when $\delta\rho = \delta k$=0.05 and 0.15 respectively. When the heterogeneity is large this technique fails for lower modes. It should be noted that even when this technique fails, it converges - i.e., if the introduced heterogeneity is too large it converges toward wrong answer.

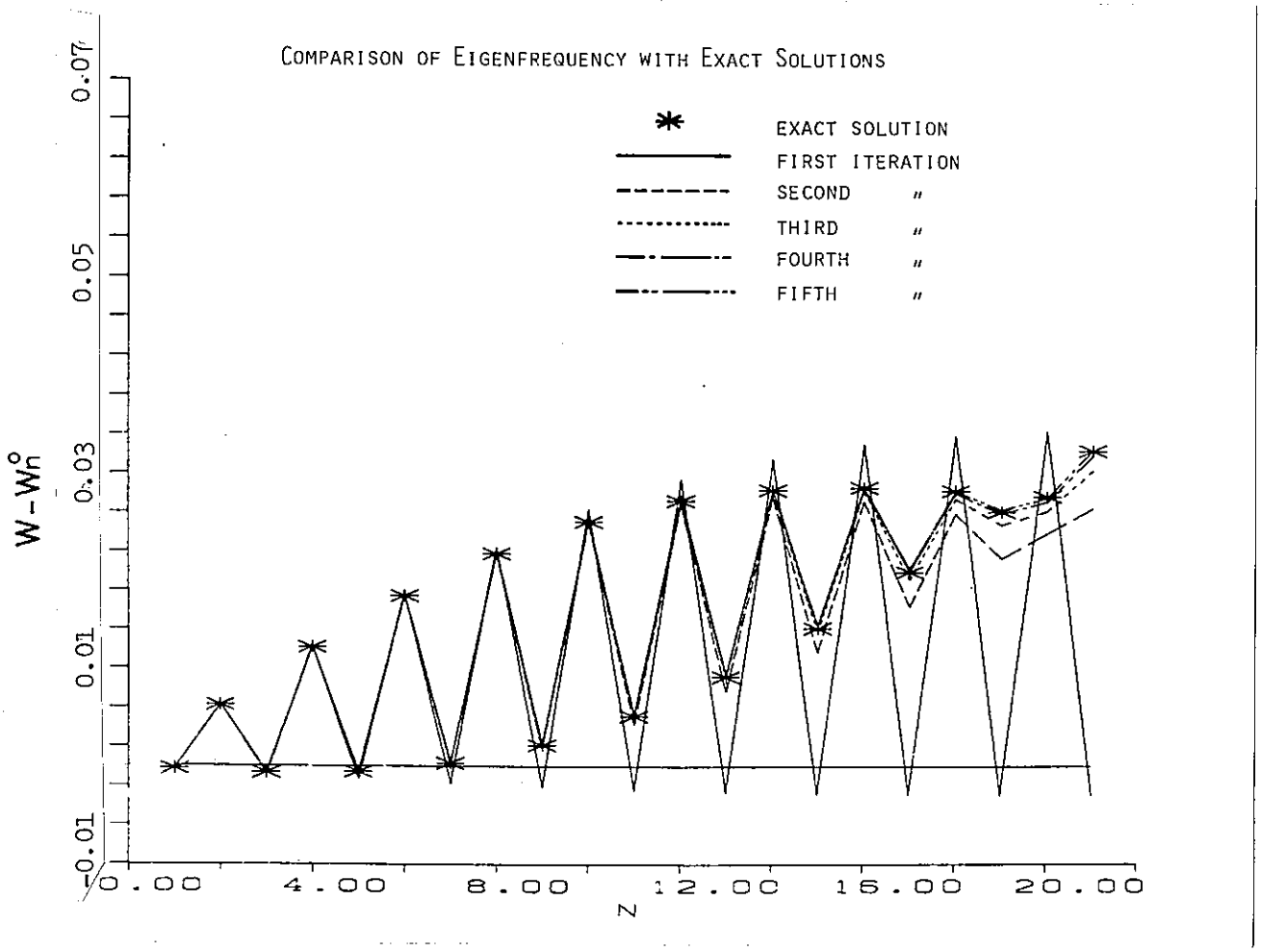

Figure 1. Comparison of eigenfrequencies for exact solution and five iterations of proposed technique. $\omega_n^{(0)}$ is subtracted from all eigenfrequencies. The horizontal solid line represents $\omega_n^{(0)}$. Four iterations seem to be enough.

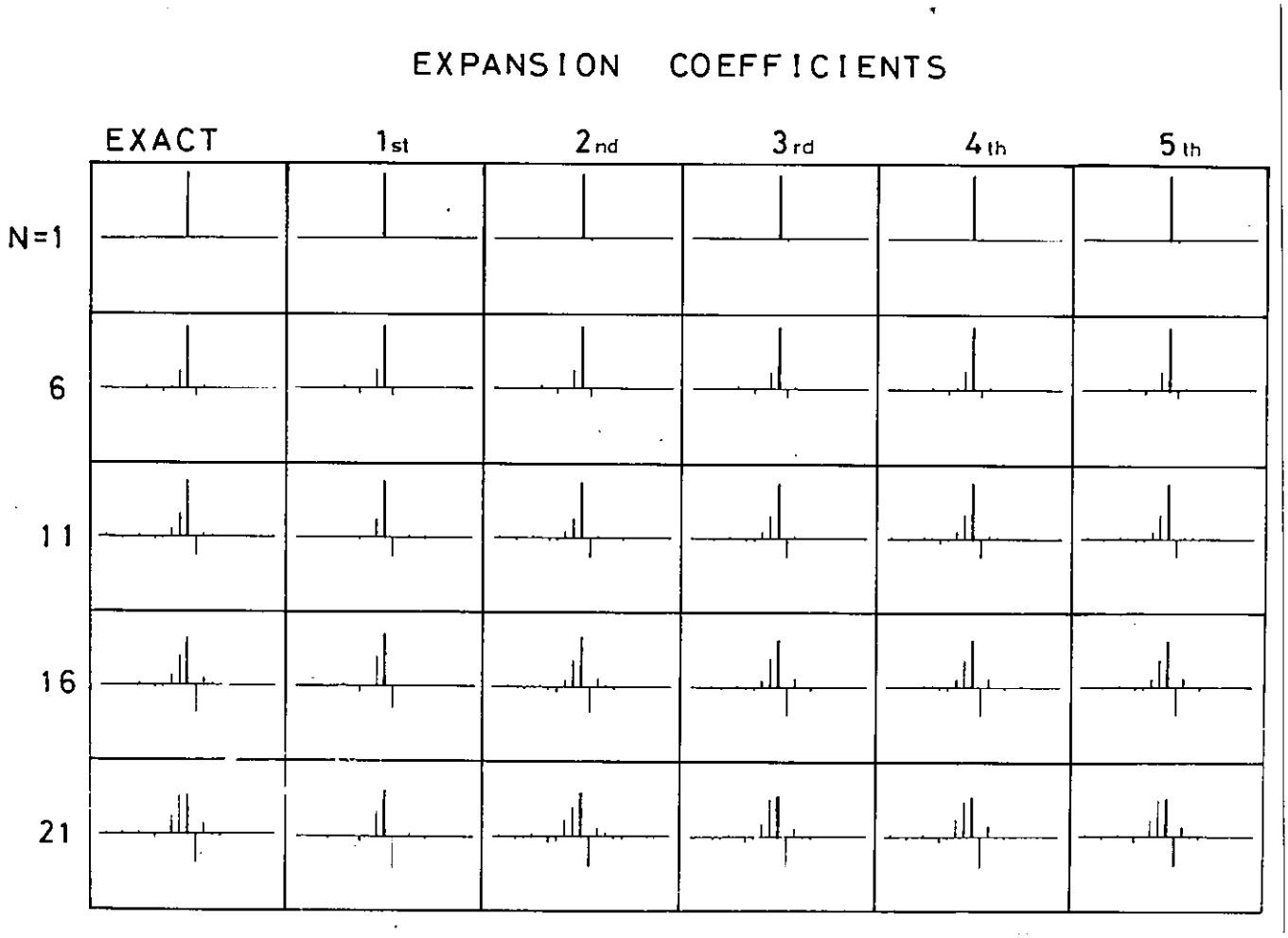

Figure 2. Comparison of expansion coefficients for exact solution and five iterations of proposed technique, for modes of order N. In each case the coefficient for Nth mode is in the center.

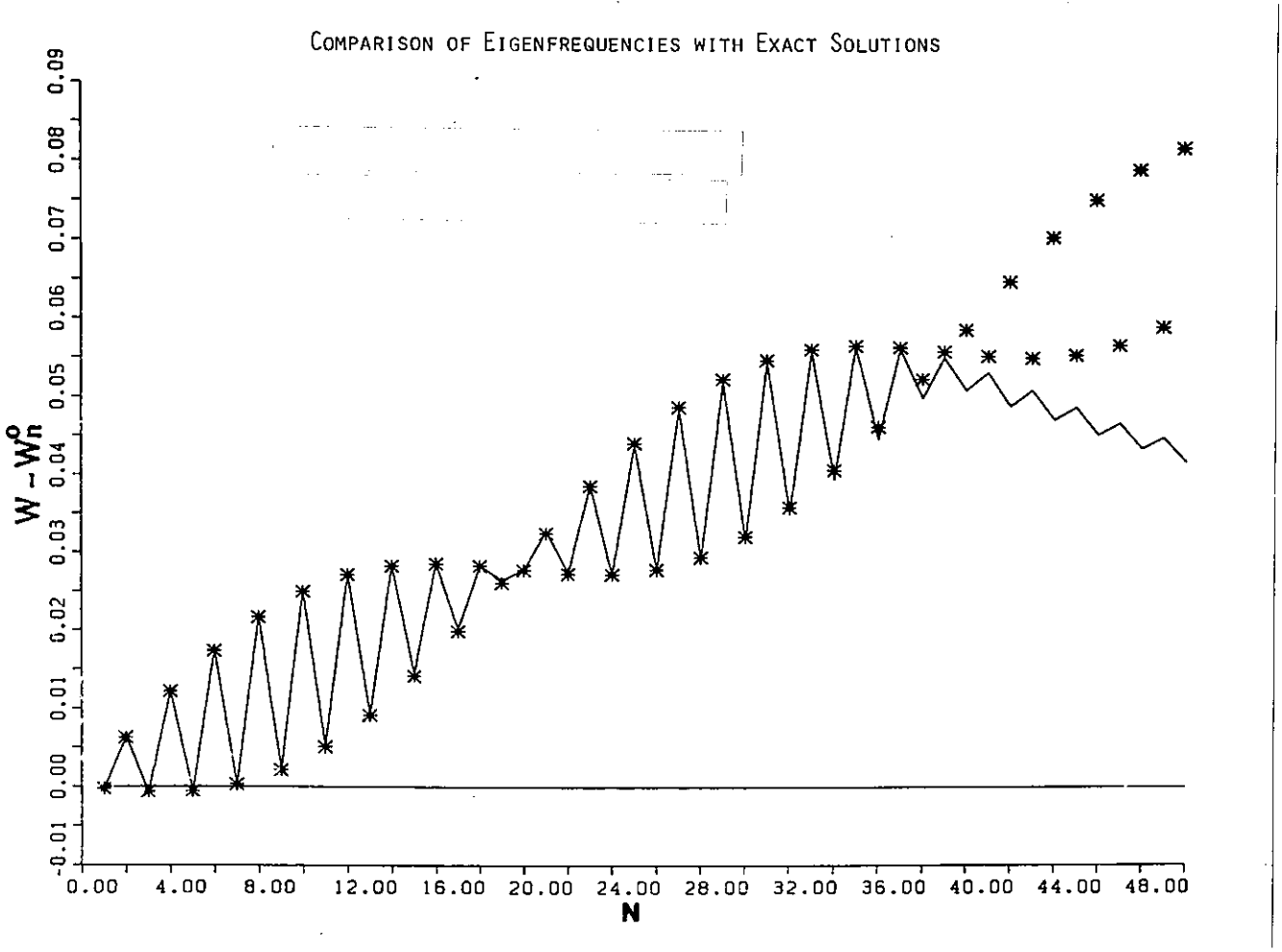

Figure 3-a. Comparison of eigenfrequencies for exact solution and proposed technique. Modes are extended to N=50. $\delta\rho=\delta k=0.1$. It starts to fail about N=40.

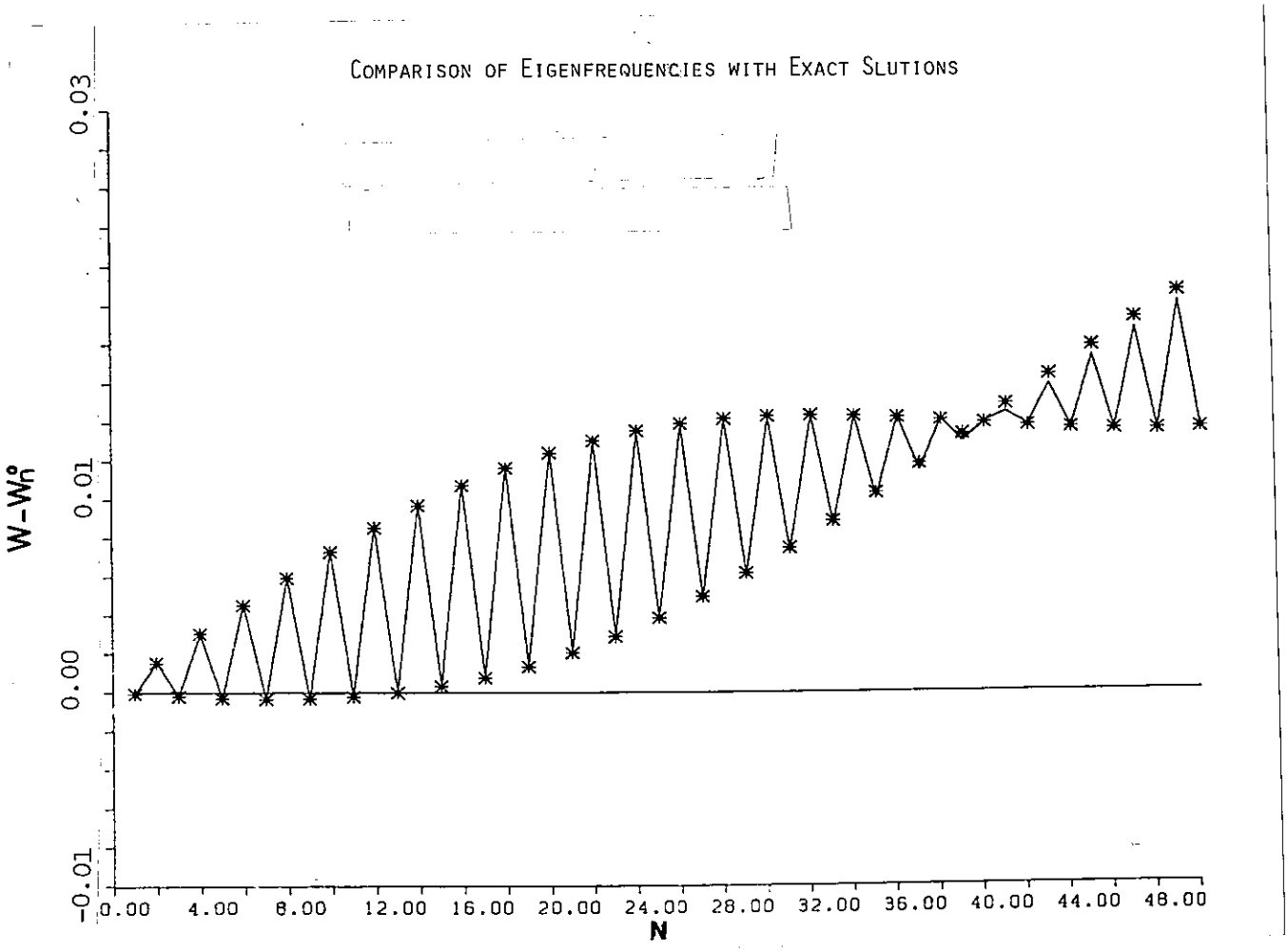

Figure 3-b. We set $\delta\rho=\delta k=0.05$. Present technique gives almost right answer.

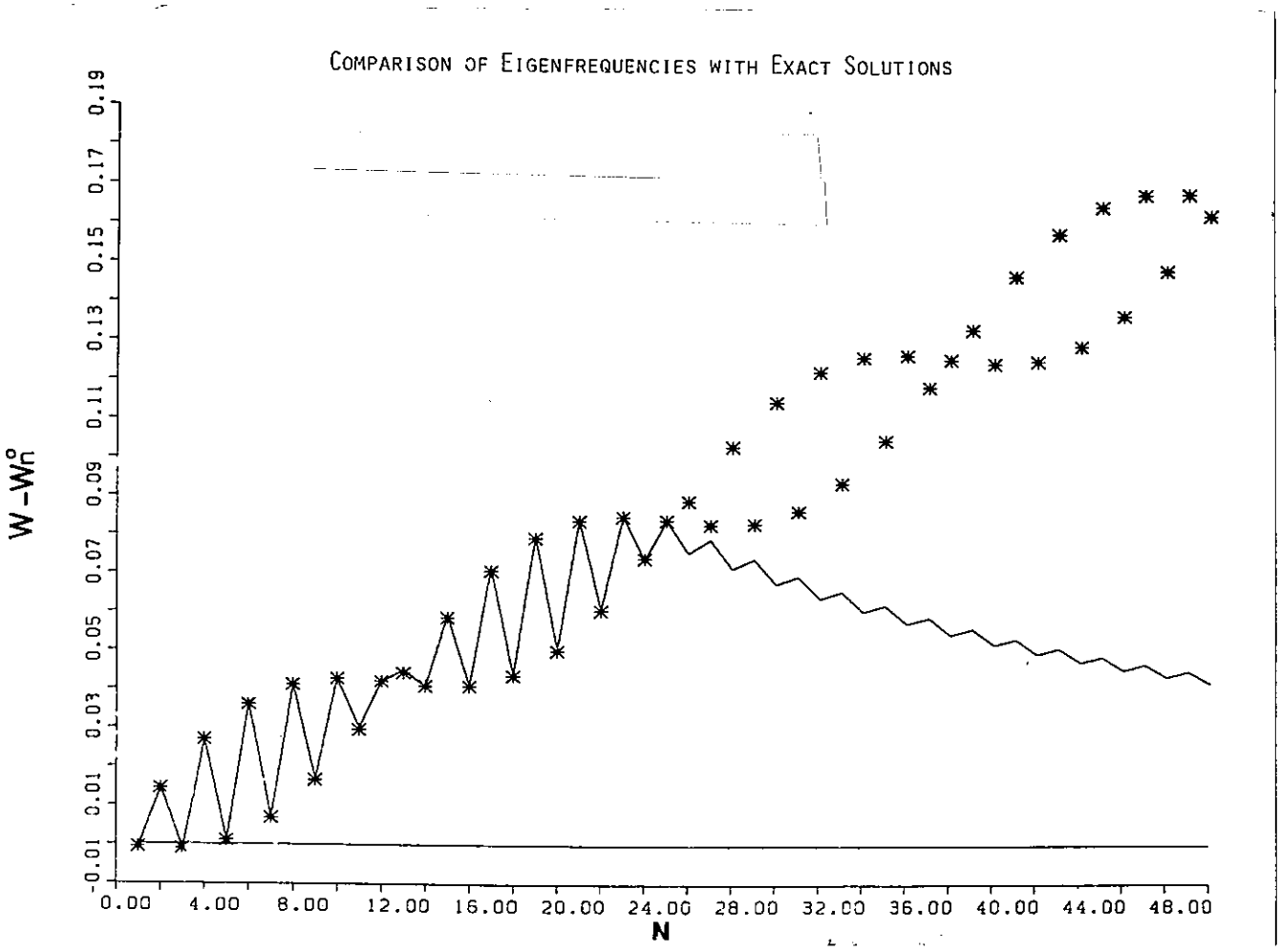

Figure 3-c. $\delta\rho=\delta k=0.15$. It begins to fail about N=26.

4.2 Heterogeneous membrane

The case when the original eigenvalue problem has some kind of degeneracy, proposed technique can not be used because the denominator of (6) will be zero. In this section we consider a heterogeneous square membrane with uniform tention, T, and variable density, ρ, which has degenerate eigenfunctions when it is homogeneous.

The Fourier transformed equation of motion is

$$T \left(\frac{\partial^2 \Psi(x,y)}{\partial x^2} + \frac{\partial^2 \Psi(x,y)}{\partial y^2} \right) + \rho(x,y)\ \omega^2 \Psi(x,y) = 0 \qquad (9)$$

with boundary conditions

$$\Psi(x,y) = 0 \qquad \text{at all boundaries.}$$

If $\rho=\rho_0$ is constant, then eigenfunctions have degeneracies, i.e., unperturbed eigenfunction of mode (n_1,n_2),

$$\Psi_n^{(0)} = \frac{2}{\sqrt{\rho_0}\, a} \sin \frac{n_1 \pi x}{a} \sin \frac{n_2 \pi y}{a}$$

where a is the length of the sides, and that of mode (n_2,n_1) have same eigenfrequency,

$$(\omega_n^{(0)})^2 = \frac{T}{\rho_0} \pi^2 \frac{n_1^2 + n_2^2}{a^2} \quad .$$

Any linear combination of these two eigenfunctions also have the same eigenfrequency. In this problem mode number,n, denotes

a pair of natural numbers, (n_1, n_2), where n_1 is a mode number of x-direction and n_2 is that of y-direction.

A perturbation of density is introduced by

$$\rho(x,y) = \rho_0 + (\delta\rho)\ x\ y\ /\ a^2 \tag{10}$$

where ρ_0 is the unperturbed density, while tention,T, is kept constant. Coupling matrices of such a problem are

$$\rho^{(1)}_{mn} = \frac{4\ \delta\rho}{\rho_0\ \pi^4}\left[\left(\frac{\sin\frac{m_1+n_1}{2}\pi}{m_1+n_1}\right)^2 - \left(\frac{\sin\frac{m_1-n_1}{2}\pi}{m_1-n_1}\right)^2\right] * \left[\left(\frac{\sin\frac{m_2+n_2}{2}\pi}{m_2+n_2}\right)^2 - \left(\frac{\sin\frac{m_2-n_2}{2}\pi}{m_2-n_2}\right)^2\right]$$

$$H^{(1)}_{mn} = 0 \tag{11}$$

where $n=(n_1,n_2)$ and $m=(m_1,m_2)$.

As mentioned earlier, when the original unperturbed eigenfunctions have degeneracy, we have to remove the degeneracy before applying this technique. First-order degenerate perturbation theory, which is common in Quantum mechanics (e.g., Schiff), is used to each degenerate multiplet.

First-order degenerate perturbation theory We consider only the effect of coupling between modes within a single multiplet - i.e., the coupling between $\Psi^{(0)}_{(n_1,n_2)}$ and $\Psi^{(0)}_{(n_2,n_1)}$ in this case. The new eigenfunction is assumed to be a linear combination of the unperturbed eigenfunctions corresponding to $(\omega^{(0)}_n)^2$.

$$\Psi = C^1 \Psi^{(0)}_{(n_1,n_2)} + C^2 \Psi^{(0)}_{(n_2,n_1)}$$

Substituting this expression into the equation of motion, (1) we have

$$\sum_{m=1}^{2} C^m (H^{(1)} + (\omega_n^{(0)})^2 \rho^{(1)}) \Psi_m^{(0)} = \rho_0 \Sigma C^m ((\omega_m)^2 - (\omega_n^{(0)})^2)$$

Only first order terms in the perturbations have been kept. The orthogonality property of the unperturbed eigenfunctions are then employed to simplify this to the final eigenvalue problem:

$$(\delta H)\ \mathbb{C} = (\omega^{(1)})^2\ \mathbb{C}$$

where $\omega^{(1)}$ is the first order correction for eigenfrequency,

$$(\delta H)_{ij} = -H_{ij}^{(1)} - (\omega^{(0)})^2\ \rho_{ij}^{(1)}$$

and

$$\mathbb{C} = (\ C^1, C^2\ )^t$$

Before applying the proposed technique, following unitary transformations are employed to get new coupling matrices corresponding to the new non-degenerate eigenfunctions.

$$H_{new}^{(1)} = C^t\ H^{(1)}\ C$$

$$\rho_{new}^{(1)} = C^t\ \rho^{(1)}\ C$$

$$C = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & 0 & & \\ & & 1 & & & & \\ & & & C^1 C^{1\prime} & & & \\ & & & C^2 C^{2\prime} & & & \\ & & & & 1 & & \\ & 0 & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

where (C^1, C^2) and ($C^{1\prime}, C^{2\prime}$) are the eigenvectors of first-order theory. And finally we use the technique including other multiplets.

Numerical results We set a=20, T=1.0 and ρ_0=0.5 and tried four cases for $\delta\rho$, $\delta\rho$= 0.1, 0.5, 1.0 and 1.5. Because of the simplicity of our example and selection rules, some of the degenerate multiplets could not be resolved by first-order perturbation theory. In Figure 4 through 7 we excluded such modes and used first 41 trial functions to solve the problem for first 21 modes, where mode number,N, was counted in the order of increasing unperturbed eigenfrequencies, $\omega_n^{(0)}$. In all figures, stars, solid-lines and broken-lines show the answers obtained by Rayleigh-Ritz method, this technique and first-order perturbation theory respectively. Figure a's of them show eigenfrequencies and b's show deviation from original unperturbed eigenfrequencies. Horizontal solid lines in Figure b's are the original eigenfrequencies.

They show that basically our technique gives almost identical eigenfrequencies with those of Rayleigh-Ritz method,

while first-order theory fails as perturbations increase. In Figure 4-a, there can not be seen any difference among three methods. In Figure 4-b, however, first-order theory deviate from Rayleigh-Ritz method and present technique only slightly. As we increase perturbations, Figure 5 - 7, the discrepancy between first-order theory and Rayleigh-Ritz method can be found even in Figure a's.

In Figure 6-b the eigenfrequency of mode N=15 looks poor. This is because eigenfrequencies of mode N=14 and 15 are nearly equal. In such a case (a kind of quasi-degenerate case) present technique may not converge to right answer. A more interesting phenominon can be seen in Figure 7-a, where apparently present technique fails for the modes indicated by arrows, N=15,17,18 and 20. But this is not true. Because we counted mode number in the order of increasing new perturbed eigenfrequencies, it would not give right mode number if eigenfrequencies of splitting multiplets exceed that of neiboring modes. If we compare the eigenfrequency of mode N=15 of Rayleigh-Ritz method with that of N=17 of present technique and so on, we see present technique gives right answer. In Figure 7-b, this is not true because the original unperturbed eigenfrequencies, which we subtract from new eigenfrequencies, are different.

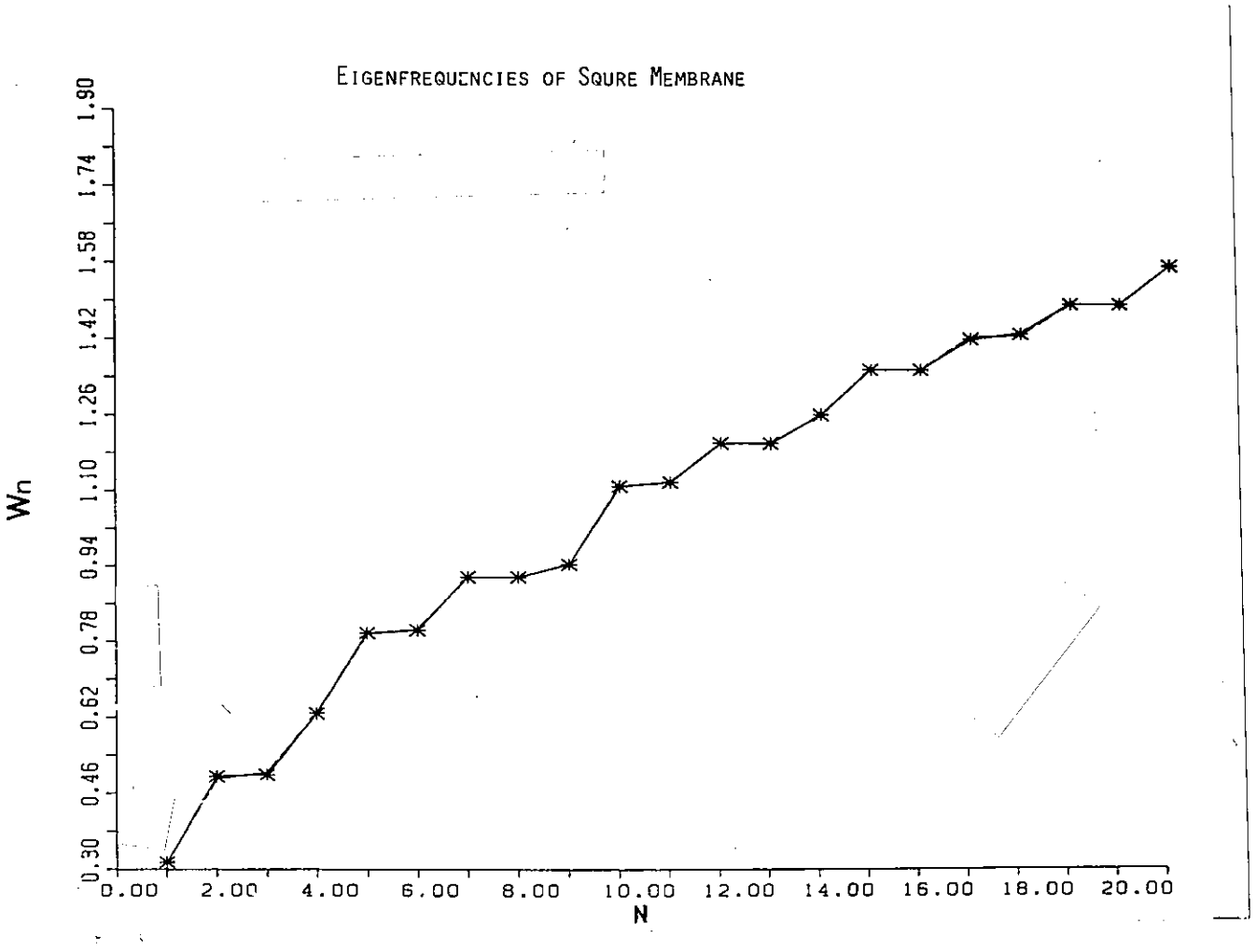

Figure 4-a. Comparison of eigenfrequencies for Rayleigh-Ritz method(*), first-order perturbation theory(broken line) and present technique(solid line). Difference among three methods can not be seen. Density change($\delta\rho/\rho_0$) is 20%.

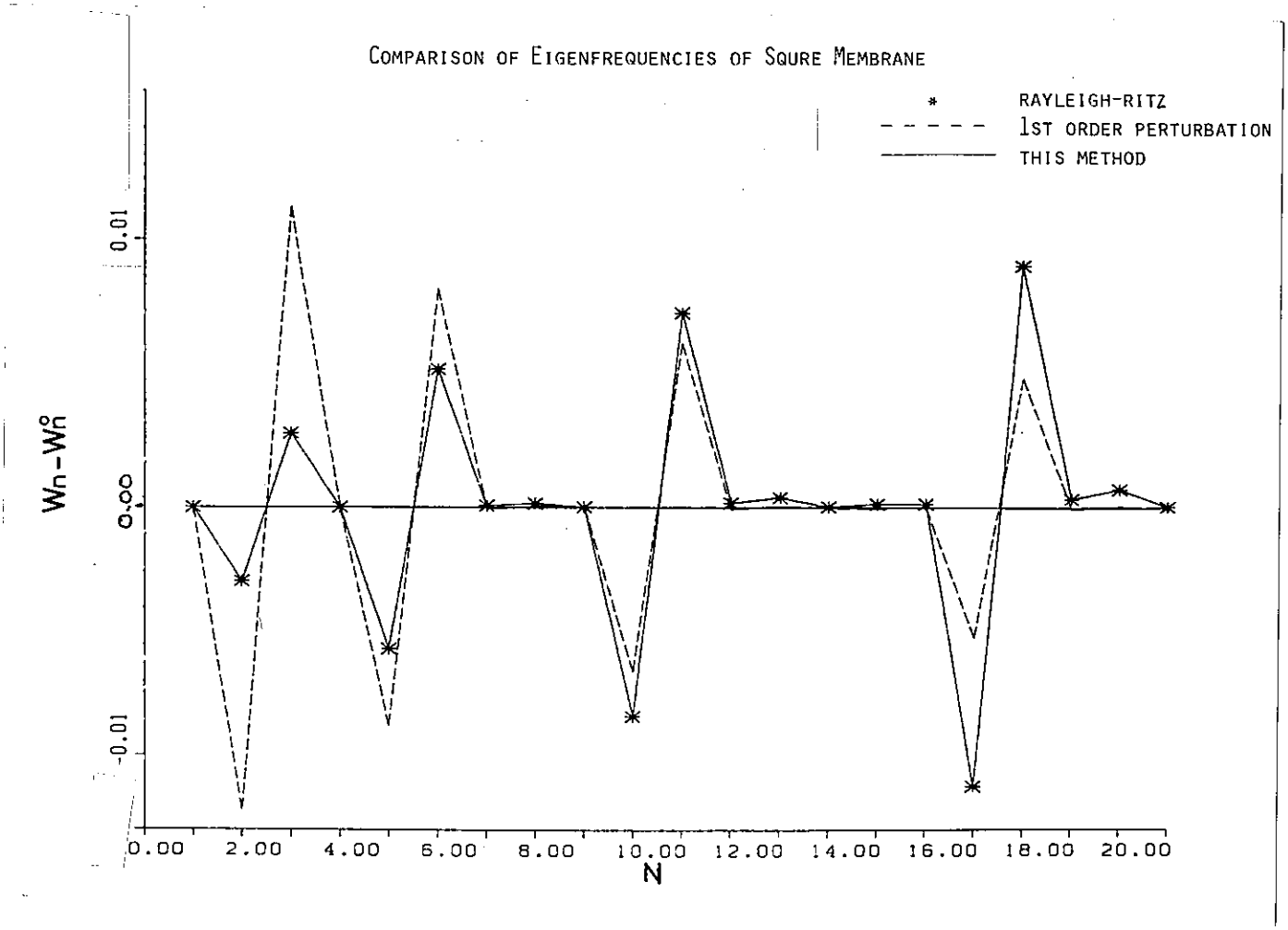

Figure 4-b. Comparison of eigenfrequencies. $\omega_n^{(0)}$ is subtracted from all eigenfrequencies. Horizontal line is the original unperturbed eigenfrequencies. $(\delta\rho/\rho_0) = 20\%$.

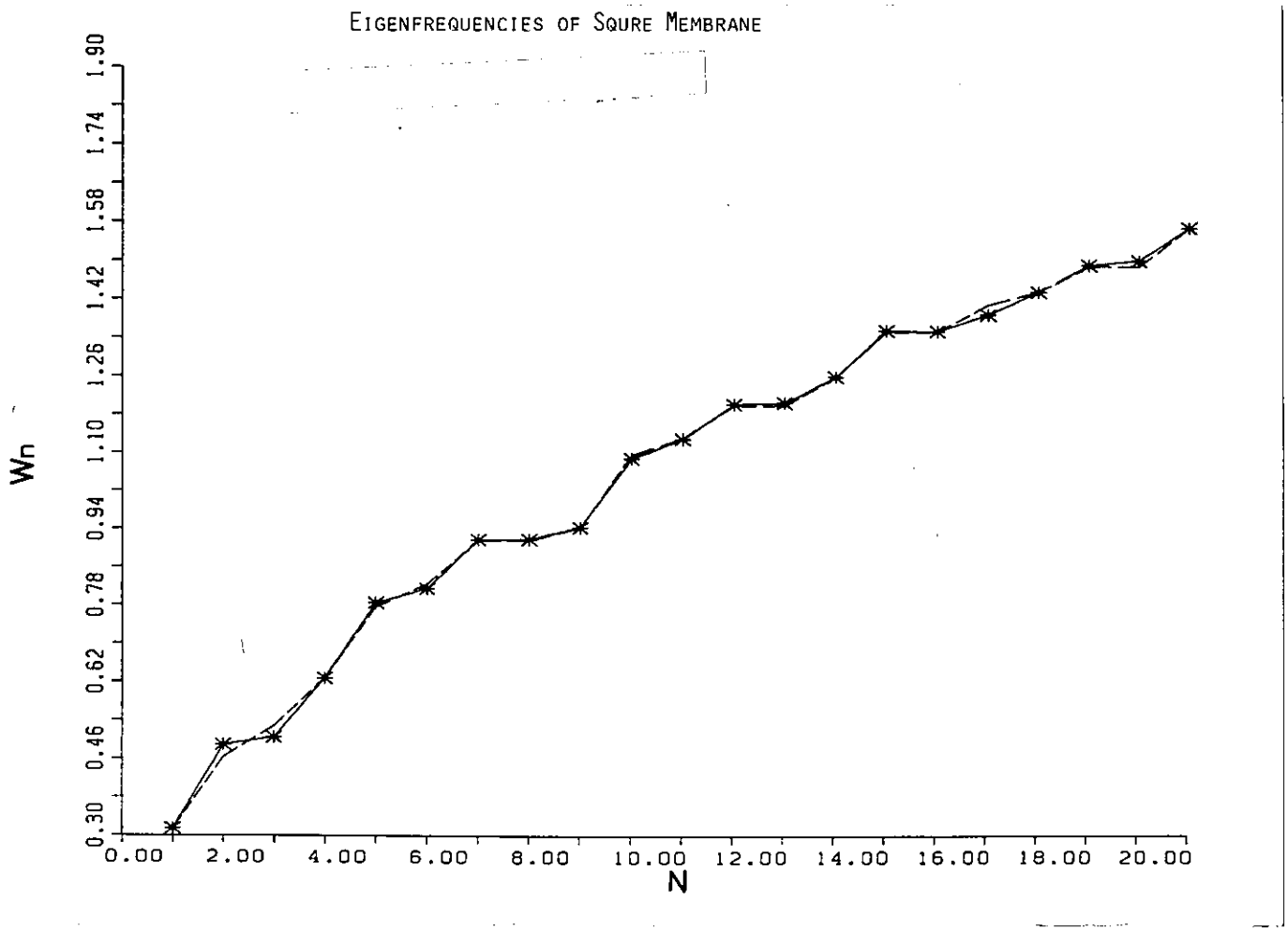

Figure 5-a. Same as Fig.4-a. $(\delta\rho/\rho_0)$=100%.

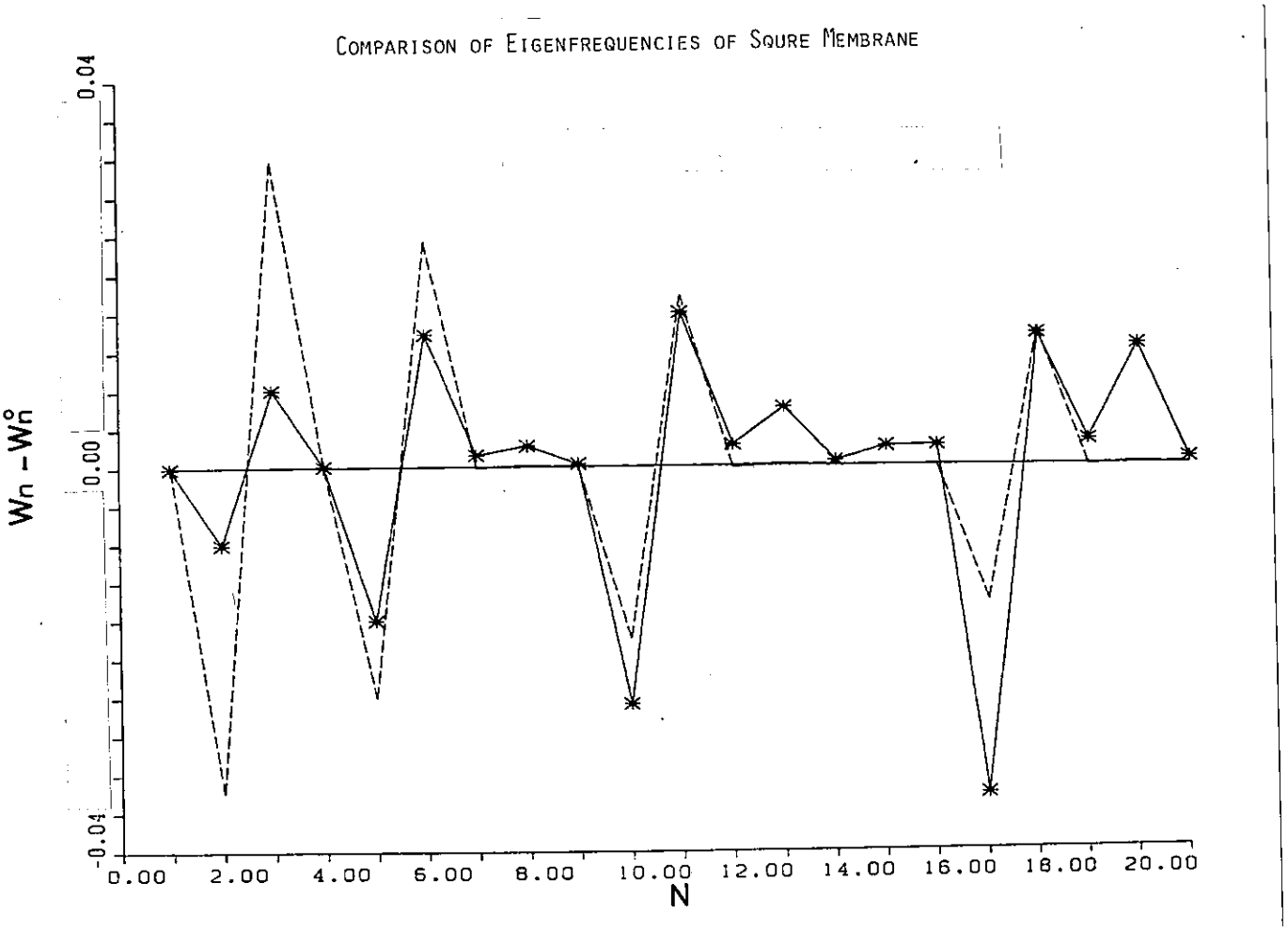

Figure 5-b. Same as Fig.4-b. $(\delta\rho/\rho_0)$=100%.

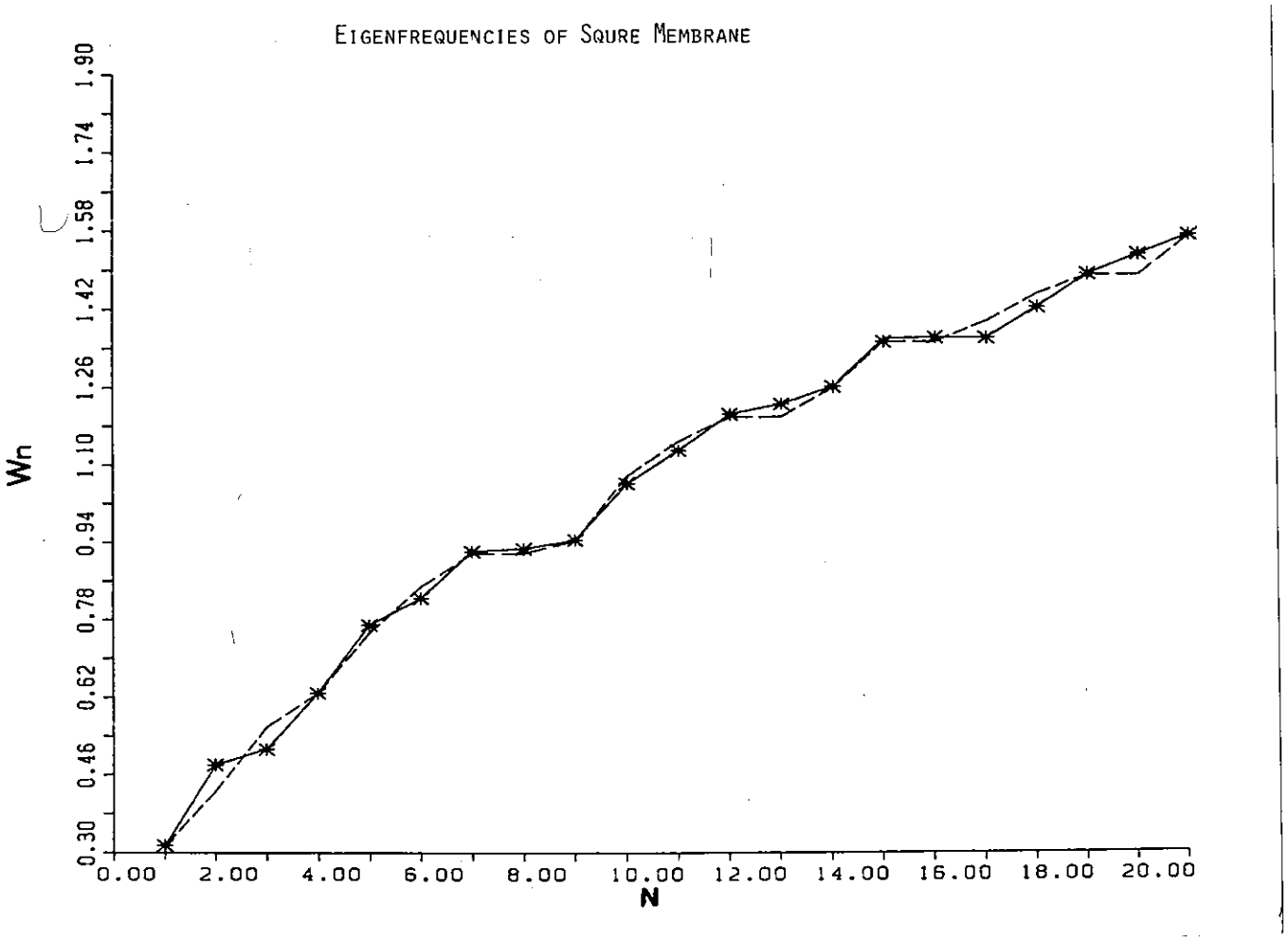

Figure 6-a. Same as Fig.4-a. $(\delta\rho/\rho_0)$= 200%. First-order theory fails apparently.

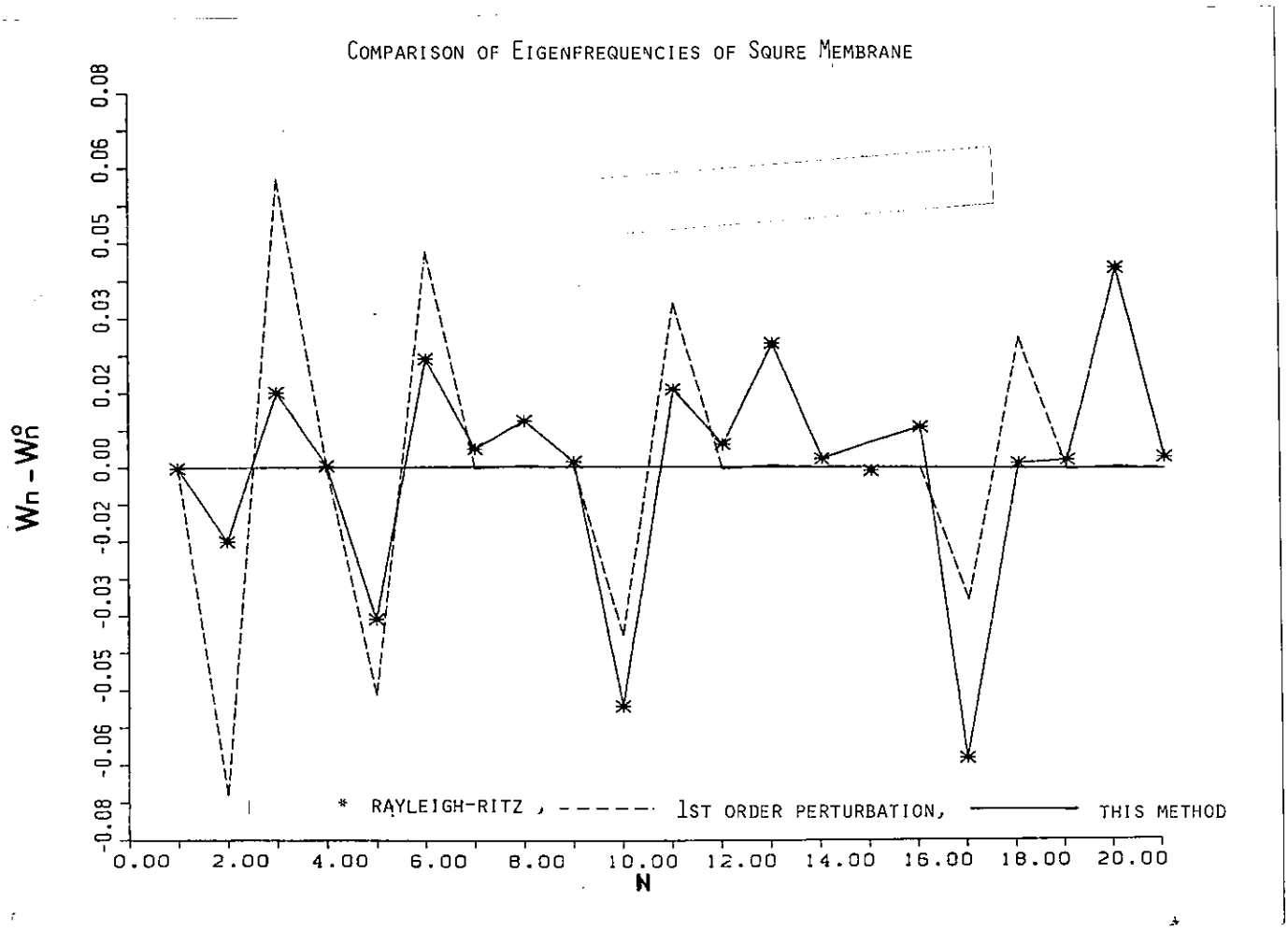

Figure 6-b. Same as Fig.4-b. $(\delta\rho/\rho_0)$=200%. Present technique is poor for N=15.

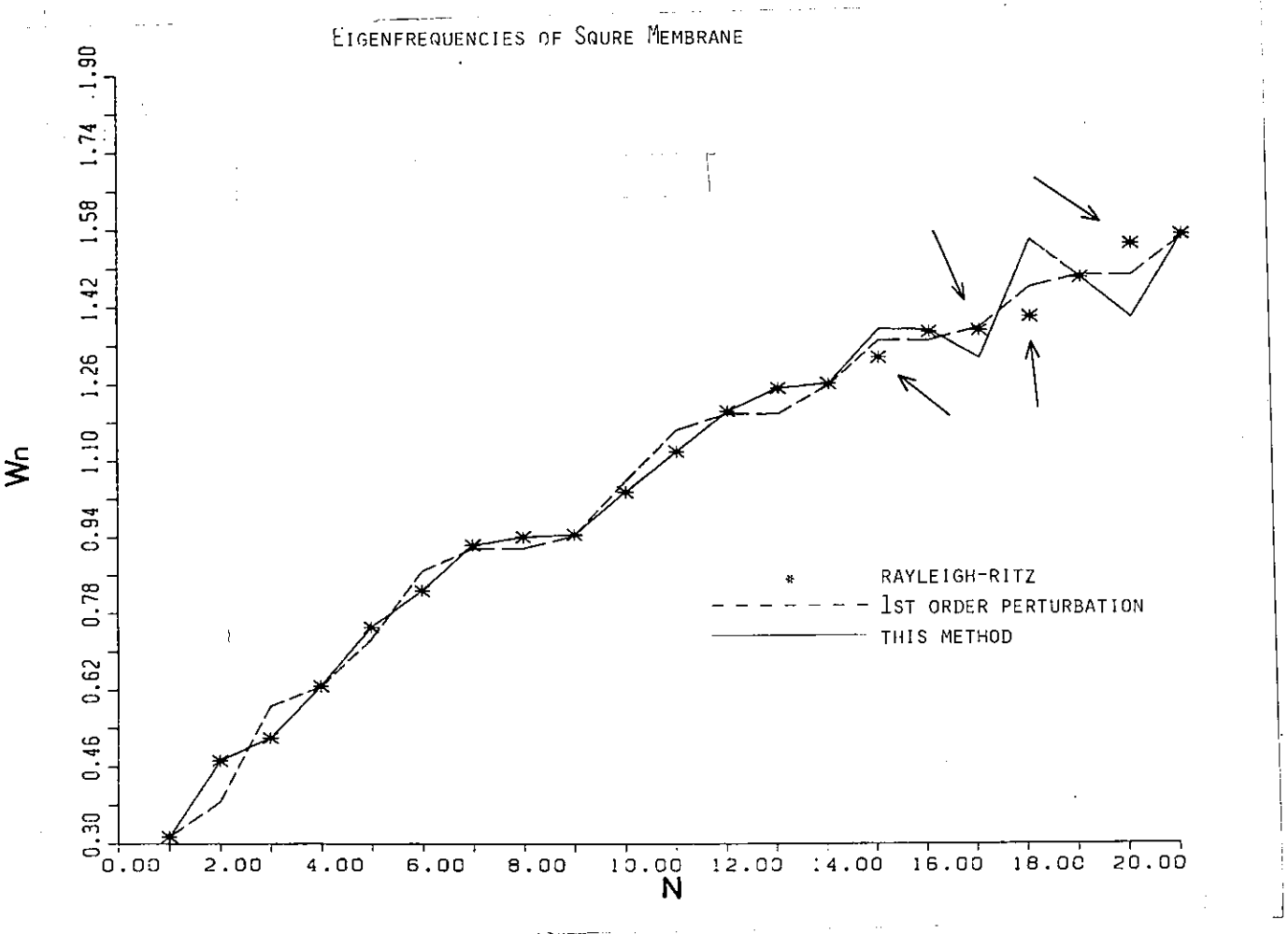

Figure 7-a. Same as Fig.4-a. $(\delta\rho/\rho_0)$=300%. For the modes indicated by arrow, present technique apparentry fails to give right solution. But it is not true. See the contex for detail.

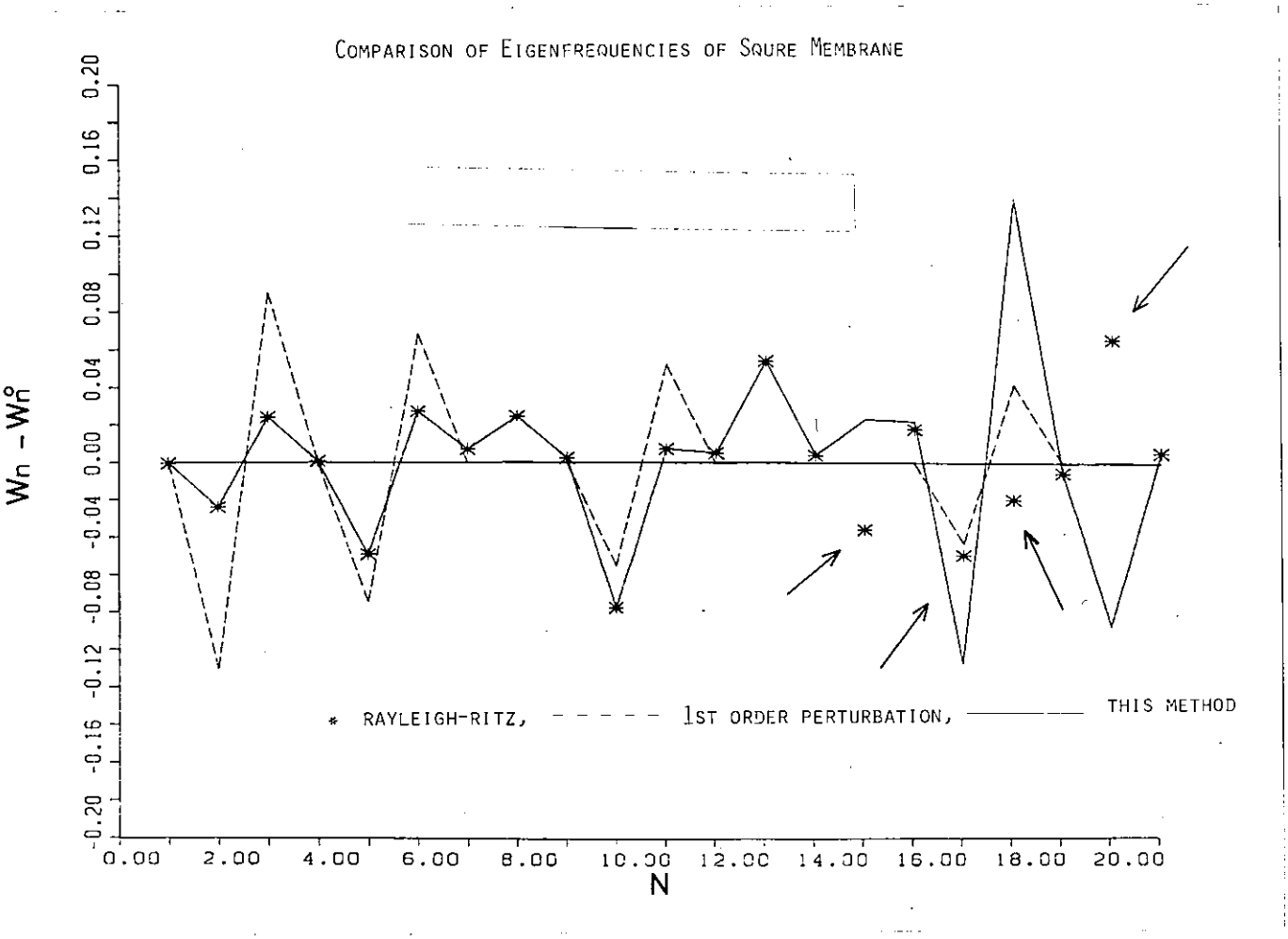

Figure 7-b. Same as Fig.4-b. $(\delta\rho/\rho_0)$=300%.

5 Discussions

In last section we showed that the proposed technique could be used for a degenerate case, if we remove degeneracy by first-order perturbation theory in advance. When the original unperturbed problem has some quasi-degenerate eigenfunctions, like last example, quasi-degenerate perturbation theory should be used instead of ordinary first-order perturbation theory.

We have introduced present new technique as the extension of second-order perturbation theory. However, another interpretation of the technique is possible from (3),(4),(5) and (6). In (6), except for C_{nn},actually we are solving set of linear simultaneous equations by a procedure similar to Jacobi method. From (4), the matrix eigenvalue problem of Rayleigh-Ritz method is written as

$$(-V + \omega_n^2 T) \mathbb{C}_n = 0.$$

If we have good approximations to eigenfrequencies, by the analogy of Jacobi method, present technique will converge toward a solution $\mathbb{C}_n=(C_{n1}, C_{n2},.....,C_{nm},.....)^t$. The residual of this solution is given by

$$(-V + \omega_n^2 T) \mathbb{C}_n = (0,.....,0,\delta_n,0,.....,0)^t$$

where

$$\delta_n = \sum_{i\neq n} (H_{ni}^{(1)} + \omega_n^2 \rho_{ni}^{(1)}) C_{ni} + (\omega_n^2 - (\omega_n^{(0)})^2) C_{nn}$$

If $\delta_n=0$, then $\mathbb{C}_n$ is the real eigenvector. So δ_n will be a good criterion of correctness of the technique.

Since the condition that Jacobi method coverges to right answer is that the specific matrix is diagonally dominant, the necessary condition that present technique works is that the matrix, $(-V+\omega_n^2 T)$, is diagonally dominant. This implies in our problem that heterogeneities are small and that $(\omega_n - \omega_m^{(0)})$ are not too small. The reason why eigenfrequency of mode N=15 of Figure 6-b is poor is that for the mode the latter condition is not fulfiled. If we use the procedure similar to that of Gauss-Sidel method instead of Jacobi method, above conditions will not be needed, because the convergence condition of Gauss-Sidel method is that the matrix is possitive-definite symetric one - so that Quasi-degenerate treatment may not be needed any more.

The fact that proposed technique converges to wrong answer in Figure 3 suggests that this technique is perturbational rather than iterative.

For the earth, what we do is:

(1) split degenerate multiplets with first-order degenerate perturbation theory,

(2) use this proposed technique including other multiplets.

6 Conclusions

As shown by Plumlee & Geller (1980), first-order degenerate perturbation theory is inadequate for dealing with lateral heterogeneity of the earth, because it loses all information about the odd harmonic order heterogeneities in an earth model. Rayleigh-Ritz method, on the other hand, does contain this information and gives good approximation to the solution if an adequate number of modes is taken into account. There are, however, serious difficulties of this method in the actual computation - solving large eigenvalue problems.

In this thesis I have described a new computational technique for calculation of normal modes of a laterally heterogeneous body. This technique has the advantages of being able to solve normal mode problem only refering to the specific mode which is to be solved. When the heterogeneity is small enough, this technique gives right answer with equivalent accuracy to Rayleigh-Ritz method.

References

Dahlen, F. A. (1969) The normal modes of a rotating, elliptical earth - II. Near-resonance multiplet coupling, Geophys. J., 18, 397-436.

Geller, R. J. and S. Stein, (1978) Normal modes of a laterally heterogeneous body: a one dimensional example, Bull. Seism. Soc. Amer., 68, 103-116.

Luh, P. C. (1973) Free oscillations of the laterally inhomgeneous earth: quasi-degenerate multiplet coupling, Geophys. J., 32, 187-202.

Luh, P. C. (1974) Normal modes of a rotating self-gravitating inhomogeneous earth, Geophys. J., 38, 187-224.

Madariaga, R. (1972) Toroidal free oscillations of the laterally heterogeneous earth, Geophys. J., 27, 81-100.

Plumlee, S. L. and R. J. Geller, (1980) A variational calculation of the toroidal modes of a simple laterally heterogeneous model, preprint.

Saito, M. (1971) Theory for the elastic-gravitational oscillation of a laterally heterogeneous earth, J. Phys. Earth, 19, 259-270.

Stifler J. F. and B. A. Bolt, (1981) Eigenvibrations of laterally inhomogeneous Earth models, Geophys. J., 64, 201-221.

Usami, T., (1971) Effect of horizontal heterogeneity of the torsional oscillation of an elastic sphere, J. Phys. Earth, 19, 175-180.

Appendix

(After G&S)

Approximation Methods for Normal Mode Problems

The variational method and first- and second-order perturbation theory are well known methods in mathematical physics, so a brief review will suffice. The equation of motion for an elastic, nongravitating, body may be written as

$$H\Psi + \rho\omega^2\Psi = 0 \tag{1}$$

where H is the (Hermitian) operator for the elastic restoring force, ρ is the density, ω is the eigenfrequency, and Ψ is the eigenfunction. The boundary conditions are

$$\alpha\Psi + \beta\frac{\partial\Psi}{\partial n} = 0 \tag{2}$$

where α and β are arbitrary constants which may be functions of position along the boundary.

All three approximation methods assume that we start with the exact (nondegenerate) eigenvalues and eigenfunctions for the problem

$$H^{(0)}\Psi^{(0)} + \rho^{(0)}(\omega^{(0)})^2\Psi^{(0)} = 0. \tag{3}$$

For simplicity, we assume that for the boundary conditions $\alpha_0 = \alpha$ and $\beta_0 = \beta$ (i.e., the boundary conditions are the same for the perturbed and unperturbed problems). We denote the eigenvalues and eigenfunctions for (3) as $\omega_n^{(0)}$ and $\Psi_n^{(0)}$, respectively. The $\Psi_n^{(0)}$ are orthonormalized so that $\int (\Psi_m^{(0)})^*\rho^0(\Psi_n^{(0)})\,dV = \delta_{mn}$.

We now consider the details of each approximation method.

First-order perturbation theory: We write the perturbed operator and density in (1) as

$$H = H^{(0)} + H^{(1)} \tag{4}$$

and

$$\rho = \rho^{(0)} + \rho^{(1)} \tag{5}$$

where $H^{(1)}$ and $\rho^{(1)}$ represent the difference between the perturbed and unperturbed problems. We then define matrix elements

$$H_{mn}^{(1)} = \int (\Psi_m^{(0)})^* H^{(1)}(\Psi_n^{(0)})\,dV$$

and

$$\rho_{mn}^{(1)} = \int (\Psi_m^{(0)})^*\rho^{(1)}(\Psi_n^{(0)})\,dV. \tag{6}$$

(Note that $H_{mn}^{(1)} = (H_{nm}^{(1)})^*$ and $\rho_{nm}^{(1)} = (\rho_{mn}^{(1)})^*$.)

In first order perturbation theory, as commonly applied in seismology, we assume that the eigenfunction for a given mode remains the same, and calculate the "first-order" correction to the eigenvalue, $(\omega^2) = (\omega^{(1)})^2 + (\omega^{(0)})^2$.

$$(\omega_n^{(1)})^2 = -(\omega_n^{(0)})^2\rho_{nn}^{(1)} - H_{nn}^{(1)} \tag{7}$$

Second-order perturbation theory: We proceed in the same way as Merzbacher (1970), except for the extra terms due to the density perturbation. We assume that we make a *small* perturbation to the eigenfunction, which we find to first order

$$\Psi_n = \Psi_n^{(0)} + \sum_{m\neq n} C_{nm}\Psi_m^{(0)} \tag{8}$$

and a *small* second-order correction to the eigenvalue

$$\omega^2 = (\omega^{(0)})^2 + (\omega^{(1)})^2 + (\omega^{(2)})^2. \tag{9}$$

When we introduce (8) and (7) into (1) we obtain, after discarding terms which are not of first order

$$C_{nm} = \frac{H_{mn}^{(1)} + (\omega_n{}^{(0)})^2 \rho_{mn}^{(1)}}{(\omega_n{}^{(0)})^2 - (\omega_m{}^{(0)})^2}. \tag{10}$$

When we substitute (7), (8), (9), and (10) into (1) we find the second-order correction to the eigenfrequency

$$(\omega_n{}^{(2)})^2 = \sum_{m \neq n} \frac{|H_{mn}^{(1)} + (\omega_n{}^{(0)})^2 \rho_{mn}^{(1)}|^2}{(\omega_m{}^{(0)})^2 - (\omega_n{}^{(0)})^2} - (\omega_n{}^{(1)})^2 \rho_{nn}^{(1)}. \tag{11}$$

Finally, it is necessary to renormalize the new eigenfunction. Although the summations in (8) and (11) are in principle over all modes, in practice we use some finite subset of the modes for which (11) will converge. The normalization factor is then

$$P = \sum_k \sum_m C_{nk}^* C_{nm} [\rho_{km}^{(1)} + \delta_{km}] \tag{12}$$

and the normalized eigenfunction is

$$\Psi_n = \frac{1}{\sqrt{P}} (\Psi_n{}^{(0)} + \sum_{m \neq n} C_{nm} \Psi_m{}^{(0)}). \tag{13}$$

Variational method: It should be emphasized that the variational method is based on Hamilton's principle, and thus can be justified physically (*cf.* Morse and Feshbach, 1953). Hamilton's principle states that for any small change in a system, the Lagrangian, L, is stationary,

$$\delta(L) = \delta(T - V) = 0, \tag{14}$$

where T is the kinetic energy and V is the potential energy. We will use the eigenfunctions for (3), the unperturbed solution, as trial functions and write the eigenfunction as

$$\Psi = \sum C_n \Psi_n{}^{(0)}. \tag{15}$$

If we insert (15) into (14) and set the partial derivatives of L with respect to C_n to zero, we obtain a matrix eigenvalue problem

$$\det (\omega^2 T - V) = 0 \tag{16}$$

where the elements of T and V are

$$\begin{aligned} T_{ij} &= \int (\Psi_i{}^{(0)})^* \rho \Psi_j{}^{(0)} \, dV = \rho_{ij}^{(1)} + \delta_{ij} \\ V_{ij} &= -\int (\Psi_i{}^{(0)})^* H \Psi_j{}^{(0)} \, dV = -H_{ij}^{(1)} + \delta_{ij} (\omega_j{}^{(0)})^2. \end{aligned} \tag{17}$$

The eigenfrequencies are given by the eigenvalues of (16) and the eigenfunctions by (15), where the coefficients C_n are from the eigenvectors of (16).

Because the eigenfunctions for (3) are a complete set, if we used all of the unperturbed modes in our trial function, (15), we would obtain the exact eigenfrequencies and eigenvalues for the perturbed problem. In practice we must limit ourselves to a finite set and verify "experimentally" that our set is large enough. Note that, as is well known, first-order perturbation theory is essentially equivalent to the variational method with only one trial function in (15).

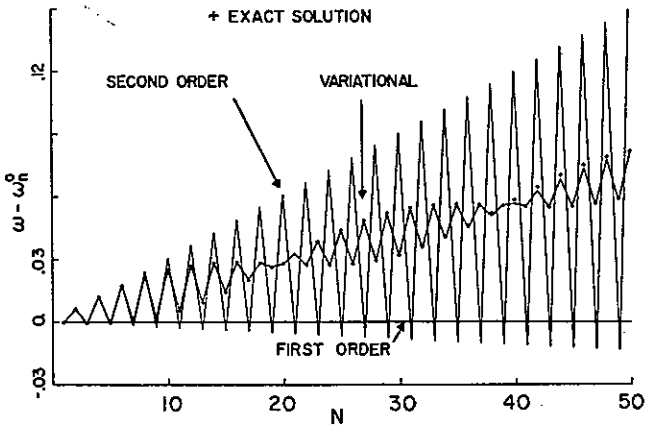

FIG. 1. Comparison of eigenfrequencies for exact solution, variational method, and first- and second-order perturbation theory. $\omega_n^{(0)}$ is subtracted from all eigenvalues for ease of presentation. The variational solution agrees excellently with the exact solution, while the second-order perturbation solution oscillates wildly around the correct asymptote. First-order perturbation yields eigenvalues which are identical to those for the unperturbed problem. To put this figure in perspective, the lowest eigenvalue is $\omega_1 = 0.2219$, so the difference between ω_{50} and $\omega_{50}^{(0)}$ is about $\frac{1}{3}$ of ω_1.

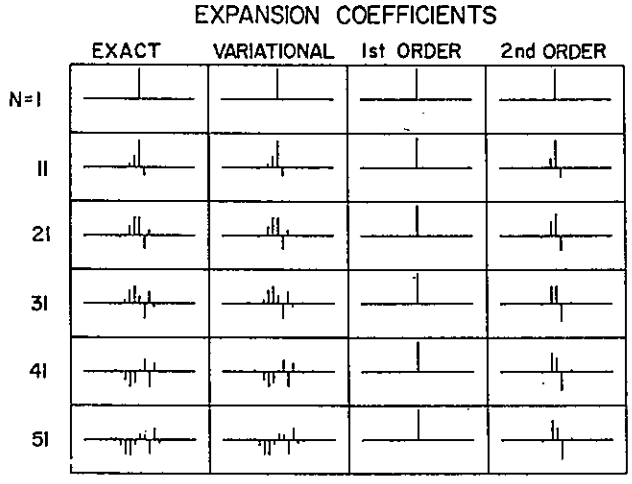

FIG. 2. Comparison of expansion coefficients for exact solution, variational method, and first- and second-order perturbation theory, for modes of order N. In each case the coefficient for the Nth mode is in the center.