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A new fifth parameter for transverse isotropy II: partial derivatives

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SUMMARY

Kawakatsu *et al.* and Kawakatsu introduced a new fifth parameter, η_κ, to describe transverse isotropy (TI). Considering that η_κ characterizes the incidence angle dependence of body wave phase velocities for TI models, its relevance for body wave seismology is obvious. Here, we derive expressions for partial derivatives (sensitivity kernels) of surface wave phase velocity and normal mode eigenfrequency for the new set of five parameters. The partial derivative for η_κ is about twice as large as that for the conventional η, indicating that η_κ should be more readily resolved. While partial derivatives for S velocities are not so changed, those for P velocities are significantly modified; the sensitivity for anisotropic P velocities is greatly reduced. In contrary to the suggestion by Dziewonski & Anderson and Anderson & Dziewonski, there is not much control on the anisotropic P velocities. On the other hand, the significance of η_κ for long-period seismology has become clear.

Key words: Surface waves and free oscillations; Seismic anisotropy; Theoretical seismology.

INTRODUCTION

Kawakatsu *et al.* (2015) and Kawakatsu (2016) recently introduced a new parameter, η_κ that characterizes the incidence angle dependence (relative to the symmetry axis) of seismic body wave velocities in a transverse isotropy (TI) system. While the commonly used fifth parameter in global seismology to describe TI system, $\eta = F/(A-2L)$ (e.g. Anderson 1961; Takeuchi & Saito 1972), has no simple physical meaning (Saito, 2015, personal communication), the newly defined parameter,

$$\eta_\kappa = \frac{F+L}{(A-L)^{1/2}(C-L)^{1/2}}, \tag{1}$$

where A, C, F and L denote Love's elastic constants for TI (Love 1927), measures the departure from the 'elliptic condition' (Thomsen 1986) when η_κ is not equal to unity, and characterizes the incidence angle dependence of body waves (Kawakatsu 2016).

Kawakatsu (2016) also showed that when existing models of upper-mantle radial anisotropy (transverse isotropy with a vertical symmetry axis, VTI) are compared in terms of η_κ, PREM (Dziewonski & Anderson 1981) shows a distinct property; within the anisotropic layer of PREM (a depth range of 24.4–220 km), $\eta_\kappa < 1$ in the upper half and $\eta_\kappa > 1$ in the lower half; if $\eta_\kappa > 1$, anisotropy cannot be attributed to the layering of homogeneous isotropic layers, and thus requires the presence of intrinsic anisotropy. This finding indicates that the introduction of η_κ at least brings a new way to interpret VTI models. Then, the next question to be answered would be how well this new set of parameters are resolved. As most of VTI models are constructed, as PREM, by analysing surface wave and/or normal mode data, to answer this question we need their partial derivatives (sensitivity kernels) for model perturbations. It should be also noted that Thomsen & Anderson (2015) recently called attention for the importance of partial derivatives to be taken with respect to appropriately chosen anisotropy parameters such that partial derivatives make physical sense.

PARTIAL DERIVATIVES

We generally follow Takeuchi & Saito (1972) for the notation and derivation of partial derivatives (see also Appendix). To describe VTI, we use two sets of anisotropy elastic parameters: $(\alpha_H, \alpha_V, \beta_V, \beta_H, \eta_\kappa)$ and $(\alpha_H, \beta_V, \xi, \phi, \eta_\kappa)$, where the two P-wave velocities and the two S-wave velocities are defined as

$$\alpha_H = \sqrt{A/\rho}$$

$$\alpha_V = \sqrt{C/\rho}$$

and

$$\beta_V = \sqrt{L/\rho}$$

$$\beta_H = \sqrt{N/\rho}$$

(ρ denotes density), and the strength of P and S anisotropy as

$$\varphi = C/A = \alpha_V^2/\alpha_H^2$$

$$\xi = N/L = \beta_H^2/\beta_V^2.$$

Formulae

Surface waves

The small change in phase velocity (c) of surface waves at a given angular frequency (ω) due to changes in material properties can be expressed as,

$$\left(\frac{\delta c}{c}\right)_\omega = \sum_i \int \frac{\epsilon_i}{c}\left[\frac{\partial c}{\partial \epsilon_i}\right]_\omega \frac{\delta \epsilon_i}{\epsilon_i}\, dz + \int \frac{\rho}{c}\left[\frac{\partial c}{\partial \rho}\right]_\omega \frac{\delta\rho}{\rho}\, dz, \tag{2}$$

where ϵ_i denotes the ith elastic parameter among the above two sets of the anisotropy parameters and ρ the density at a depth z (e.g. Aki & Richards (1980)). The introduction of η_κ has no effect for density sensitivity, $\frac{\rho}{c}\left[\frac{\partial c}{\partial \rho}\right]_\omega$. For Love waves, it also has no effect on the partial derivatives $\frac{\epsilon_i}{c}\left[\frac{\partial c}{\partial \epsilon_i}\right]_\omega$ as given in Takeuchi & Saito (1972). For Rayleigh waves, when we use the first set (α_H, α_V, β_V, β_H, η_κ) as independent parameters, we find

$$\begin{aligned}
\frac{\alpha_H}{c}\left[\frac{\partial c}{\partial \alpha_H}\right]_\omega &= \frac{1}{\omega^2 I_1}\left(\frac{c}{U}\right) kAy_3\left[ky_3 - \frac{F+L}{A-L}\dot{y}_1\right]\\
\frac{\alpha_V}{c}\left[\frac{\partial c}{\partial \alpha_V}\right]_\omega &= \frac{1}{\omega^2 I_1}\left(\frac{c}{U}\right) C\dot{y}_1\left[\dot{y}_1 - k\frac{F+L}{C-L}y_3\right]\\
\frac{\beta_V}{c}\left[\frac{\partial c}{\partial \beta_V}\right]_\omega &= \frac{1}{\omega^2 I_1}\left(\frac{c}{U}\right)\left[\frac{1}{L}y_4^2 + kL\left\{2 + \left(\frac{1}{A-L} + \frac{1}{C-L}\right)(F+L)\right\}\dot{y}_1 y_3\right]\\
\frac{\beta_H}{c}\left[\frac{\partial c}{\partial \beta_H}\right]_\omega &= 0\\
\frac{\eta_\kappa}{c}\left[\frac{\partial c}{\partial \eta_\kappa}\right]_\omega &= \frac{-1}{\omega^2 I_1}\left(\frac{c}{U}\right) k(F+L)\dot{y}_1 y_3,
\end{aligned} \tag{3}$$

where k denotes the wavenumber and U the group velocity, $y_i = y_i(z; \omega, k)$ is the ith depth-dependent eigenfunction and $\dot{y}_i = dy_i/dz$ and the energy integral

$$I_1 = \int \rho(y_1^2 + y_3^2)\, dz. \tag{4}$$

Note that the term $(F + L)$ is present in all non-zero partial derivatives. As $(F + L) = \eta_\kappa(A - L)^{1/2}(C - L)^{1/2}$, it is possible to express them in terms of η_κ, a requirement for these new formulae to be correct. In fact, it is easy to show that partial derivatives can be expressed in terms of the five chosen anisotropy parameters (and the density) without employing Love's elastic constants.

For the case of the second set of parameters (α_H, β_V, ξ, ϕ, η_κ), we find the following:

$$\begin{aligned}
\frac{\alpha_H}{c}\left[\frac{\partial c}{\partial \alpha_H}\right]_\omega^{(*)} &= \frac{1}{\omega^2 I_1}\left(\frac{c}{U}\right)\left[k^2Ay_3^2 + C\dot{y}_1^2 - k(F+L)\left(\frac{A}{A-L} + \frac{C}{C-L}\right)\dot{y}_1 y_3\right]\\
&= \frac{1}{\omega^2 I_1}\left(\frac{c}{U}\right)\left[kAy_3\left(ky_3 - \frac{F+L}{A-L}\dot{y}_1\right) + C\dot{y}_1\left(\dot{y}_1 - k\frac{F+L}{C-L}y_3\right)\right]\\
\frac{\beta_V}{c}\left[\frac{\partial c}{\partial \beta_V}\right]_\omega^{(*)} &= \frac{1}{\omega^2 I_1}\left(\frac{c}{U}\right)\left[\frac{1}{L}y_4^2 + kL\left\{2 + \left(\frac{1}{A-L} + \frac{1}{C-L}\right)(F+L)\right\}\dot{y}_1 y_3\right]\\
\frac{\xi}{c}\left[\frac{\partial c}{\partial \xi}\right]_\omega^{(*)} &= 0
\end{aligned} \tag{5}$$

$$\frac{\phi}{c}\left[\frac{\partial c}{\partial \phi}\right]_{\omega}^{(*)} = \frac{1}{2\omega^2 I_1}\left(\frac{c}{U}\right) C\dot{y}_1\left[\dot{y}_1 - k\frac{F+L}{C-L}y_3\right]$$

$$\frac{\eta_\kappa}{c}\left[\frac{\partial c}{\partial \eta_\kappa}\right]_{\omega}^{(*)} = \frac{-1}{\omega^2 I_1}\left(\frac{c}{U}\right) k(F+L)\dot{y}_1 y_3,$$

where superscript parentheses asterisks (*) on the left-hand side are introduced to denote that these partial derivatives are for the second set of parameters. The relationship between the derivatives for the two sets of parameters are given by

$$\frac{\alpha_H}{c}\left[\frac{\partial c}{\partial \alpha_H}\right]_{\omega}^{(*)} = \frac{\alpha_H}{c}\left[\frac{\partial c}{\partial \alpha_H}\right]_{\omega} + \frac{\alpha_V}{c}\left[\frac{\partial c}{\partial \alpha_V}\right]_{\omega}$$

$$\frac{\beta_V}{c}\left[\frac{\partial c}{\partial \beta_V}\right]_{\omega}^{(*)} = \frac{\beta_V}{c}\left[\frac{\partial c}{\partial \beta_V}\right]_{\omega} + \frac{\beta_H}{c}\left[\frac{\partial c}{\partial \beta_H}\right]_{\omega} \tag{6}$$

$$\frac{\xi}{c}\left[\frac{\partial c}{\partial \xi}\right]_{\omega}^{(*)} = \frac{1}{2}\frac{\beta_H}{c}\left[\frac{\partial c}{\partial \beta_H}\right]_{\omega}$$

$$\frac{\phi}{c}\left[\frac{\partial c}{\partial \phi}\right]_{\omega}^{(*)} = \frac{1}{2}\frac{\alpha_V}{c}\left[\frac{\partial c}{\partial \alpha_V}\right]_{\omega}.$$

Normal Modes

The small change in eigenfrequencies (ω_n) of normal modes for a given angular order (n) can be expressed as,

$$\left(\frac{\delta\omega}{\omega}\right)_n = \sum_i \int \frac{\epsilon_i}{\omega}\left[\frac{\partial \omega}{\partial \epsilon_i}\right]_n \frac{\delta\epsilon_i}{\epsilon_i}\,dz + \int \frac{\rho}{\omega}\left[\frac{\partial \omega}{\partial \rho}\right]_n \frac{\delta\rho}{\rho}\,dz. \tag{7}$$

As in the case for surface waves, the introduction of η_κ has no effect for the density sensitivity, as well as for toroidal oscillations. For spheroidal oscillations (without self-gravity) with independent parameters ($\alpha_H, \alpha_V, \beta_V, \beta_H, \eta_\kappa$) , we find

$$\frac{\alpha_H}{\omega}\left[\frac{\partial \omega}{\partial \alpha_H}\right]_n = \frac{1}{\omega^2 I_1}\left[A\Psi\left(\frac{F+L}{A-L}r\dot{y}_1 + \Psi\right)\right]$$

$$\frac{\alpha_V}{\omega}\left[\frac{\partial \omega}{\partial \alpha_V}\right]_n = \frac{1}{\omega^2 I_1}Cr\dot{y}_1\left[r\dot{y}_1 + \frac{F+L}{C-L}\Psi\right]$$

$$\frac{\beta_V}{\omega}\left[\frac{\partial \omega}{\partial \beta_V}\right]_n = \frac{1}{\omega^2 I_1}L\left[\frac{n(n+1)}{L^2}(ry_4)^2 - r\dot{y}_1\Psi\left\{2 + \left(\frac{1}{A-L} + \frac{1}{C-L}\right)(F+L)\right\}\right] \tag{8}$$

$$\frac{\beta_H}{\omega}\left[\frac{\partial \omega}{\partial \beta_H}\right]_n = \frac{1}{\omega^2 I_1}2N\left[n(n+1)y_3(y_1 - y_3) - y_1\Psi\right]$$

$$\frac{\eta_\kappa}{\omega}\left[\frac{\partial \omega}{\partial \eta_\kappa}\right]_n = \frac{1}{\omega^2 I_1}(F+L)r\dot{y}_1\Psi$$

where $y_i = y_i(r)$ is the ith radial eigenfunction,

$$I_1 = \int \rho[y_1^2 + n(n+1)y_3^2]r^2\,dr,$$

and

$$\Psi = 2y_1 - n(n+1)y_3.$$

For the case of independent parameters ($\alpha_H, \beta_V, \xi, \phi, \eta_\kappa$), we find the following:

$$\frac{\alpha_H}{\omega}\left[\frac{\partial \omega}{\partial \alpha_H}\right]_n^{(*)} = \frac{1}{\omega^2 I_1}\left[A\Psi\left(\frac{F+L}{A-L}r\dot{y}_1 + \Psi\right) + Cr\dot{y}_1\left(r\dot{y}_1 + \frac{F+L}{C-L}\Psi\right)\right]$$

$$= \frac{1}{\omega^2 I_1}\left[A\Psi^2 + C(r\dot{y}_1)^2 + r\dot{y}_1\Psi\left(\frac{A}{A-L} + \frac{C}{C-L}\right)(F+L)\right]$$

$$\frac{\beta_V}{\omega}\left[\frac{\partial \omega}{\partial \beta_V}\right]_n^{(*)} = \frac{1}{\omega^2 I_1}L\left[\frac{n(n+1)}{L^2}(ry_4)^2 - r\dot{y}_1\Psi\left\{2 + \left(\frac{1}{A-L} + \frac{1}{C-L}\right)(F+L)\right\}\right]$$

$$+ \frac{1}{\omega^2 I_1}2N\left[n(n+1)y_3(y_1 - y_3) - y_1\Psi\right]$$

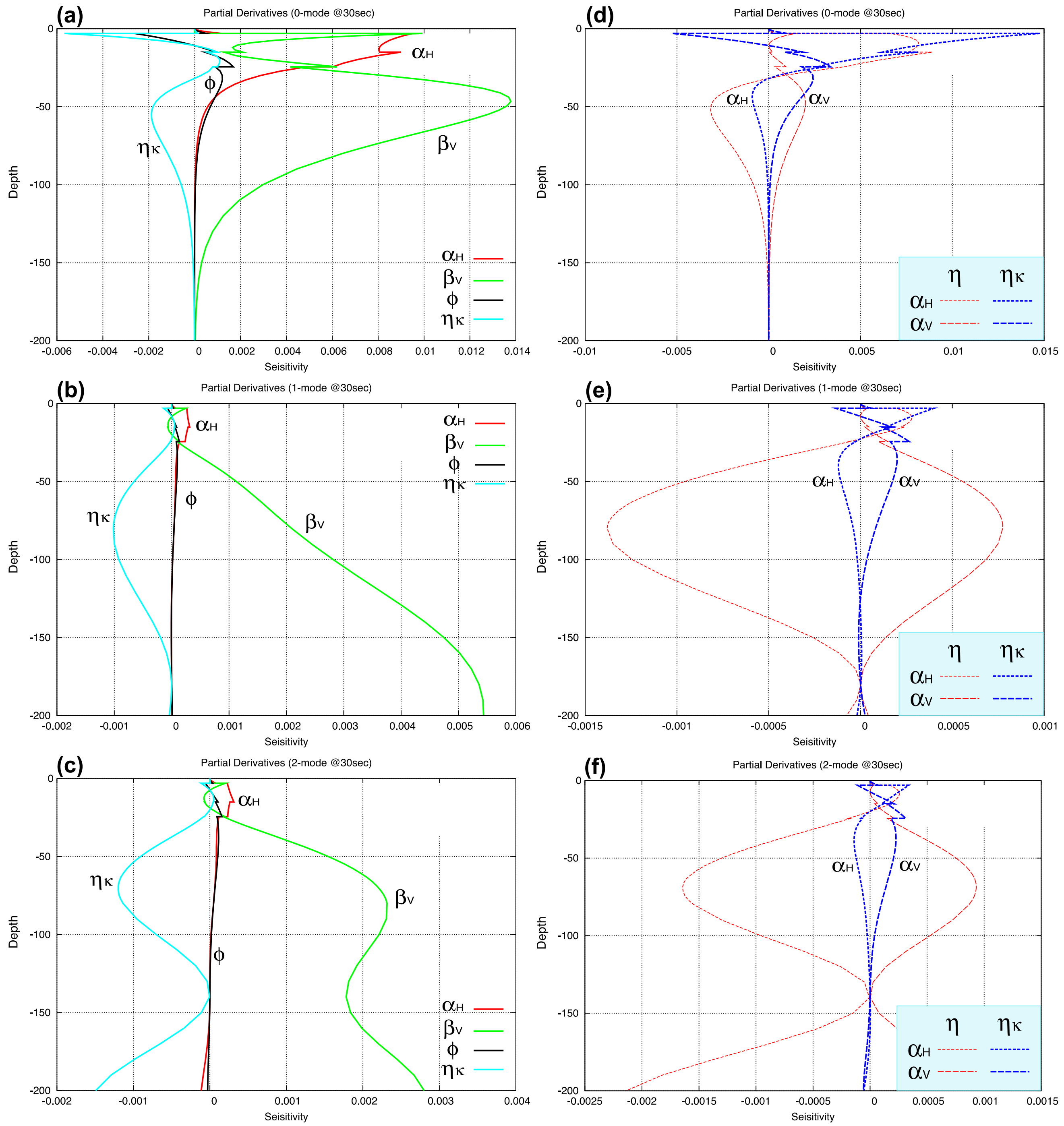

Figure 1. Partial derivatives for Rayleigh waves calculated for a flat PREM model at a period of 30 s: (a) and (d) fundamental mode surface wave, (b) and (e) first overtone surface wave and (c) and (f) second overtone surface wave. (a)–(c) are for $(\alpha_H, \beta_V, \xi, \phi, \eta_\kappa)$ as parameters. (d)–(f) compare the anisotropic *P*-wave sensitivity for $(\alpha_H, \alpha_V, \beta_V, \beta_H, \eta$ or $\eta_\kappa)$ parameter sets. Note that *P*-wave sensitivity is greatly reduced for η_κ case, especially for the over tones.

$$\frac{\xi}{\omega}\left[\frac{\partial\omega}{\partial\xi}\right]_n^{(*)} = \frac{1}{\omega^2 I_1} N\left[n(n+1)y_3(y_1 - y_3) - y_1\Psi\right] \tag{9}$$

$$\frac{\phi}{\omega}\left[\frac{\partial\omega}{\partial\phi}\right]_n^{(*)} = \frac{1}{2\omega^2 I_1} C r\dot{y}_1\left[r\dot{y}_1 + \frac{F+L}{C-L}\Psi\right]$$

$$\frac{\eta_\kappa}{\omega}\left[\frac{\partial\omega}{\partial\eta_\kappa}\right]_n^{(*)} = \frac{1}{\omega^2 I_1}(F+L) r\dot{y}_1\Psi,$$

where similar relations as in (6) hold.

Examples

Rayleigh wave

Fig. 1 shows examples of partial derivatives for Rayleigh waves calculated for a flat PREM model (Dziewonski & Anderson 1981) at a period of 30 s. The calculation is conducted using the DISPER80 program (Saito 1988) after necessary modification for the new partial derivatives. Inspection of (3) and (5), and (A1) and (A2) indicates the following relation,

$$\frac{\eta_\kappa}{c}\left[\frac{\partial c}{\partial \eta_\kappa}\right]_\omega = \frac{F+L}{F}\frac{\eta}{c}\left[\frac{\partial c}{\partial \eta}\right]_\omega. \tag{10}$$

Considering that F and L are similar in value, the sensitivity of η_κ is about twice as large as that of η. In the mantle, the sensitivity is strongest for S-wave velocity (β_V) and that for η_κ is the second largest. Figs 1(d)–(f) compare the sensitivity of anisotropic P waves. The introduction of η_κ significantly reduces the sensitivity of P waves, especially for overtones (Figs 1e and f). Considering that the overtones are related to multiple S waves, such as SS, SSS, etc., it follows that the sensitivity of η_κ is inappropriately mapped into that of $\alpha_{H,V}$ when the conventional η is used, which will be discussed further in the Discussion.

Spheroidal mode

Figs 2 and 3 show examples of partial derivatives for spheroidal modes calculated for PREM (Dziewonski & Anderson 1981). Four modes, (1) ${}_0S_{80}$, (2) ${}_1S_{40}$, (3) ${}_2S_{30}$, (4) ${}_3S_{73}$ are specifically chosen to be those investigated by Anderson & Dziewonski (1982). Fig. 2 shows the change in the anisotropic P-wave velocity partial derivatives by the introduction of η_κ. As in case of Rayleigh waves, the sensitivity of anisotropic

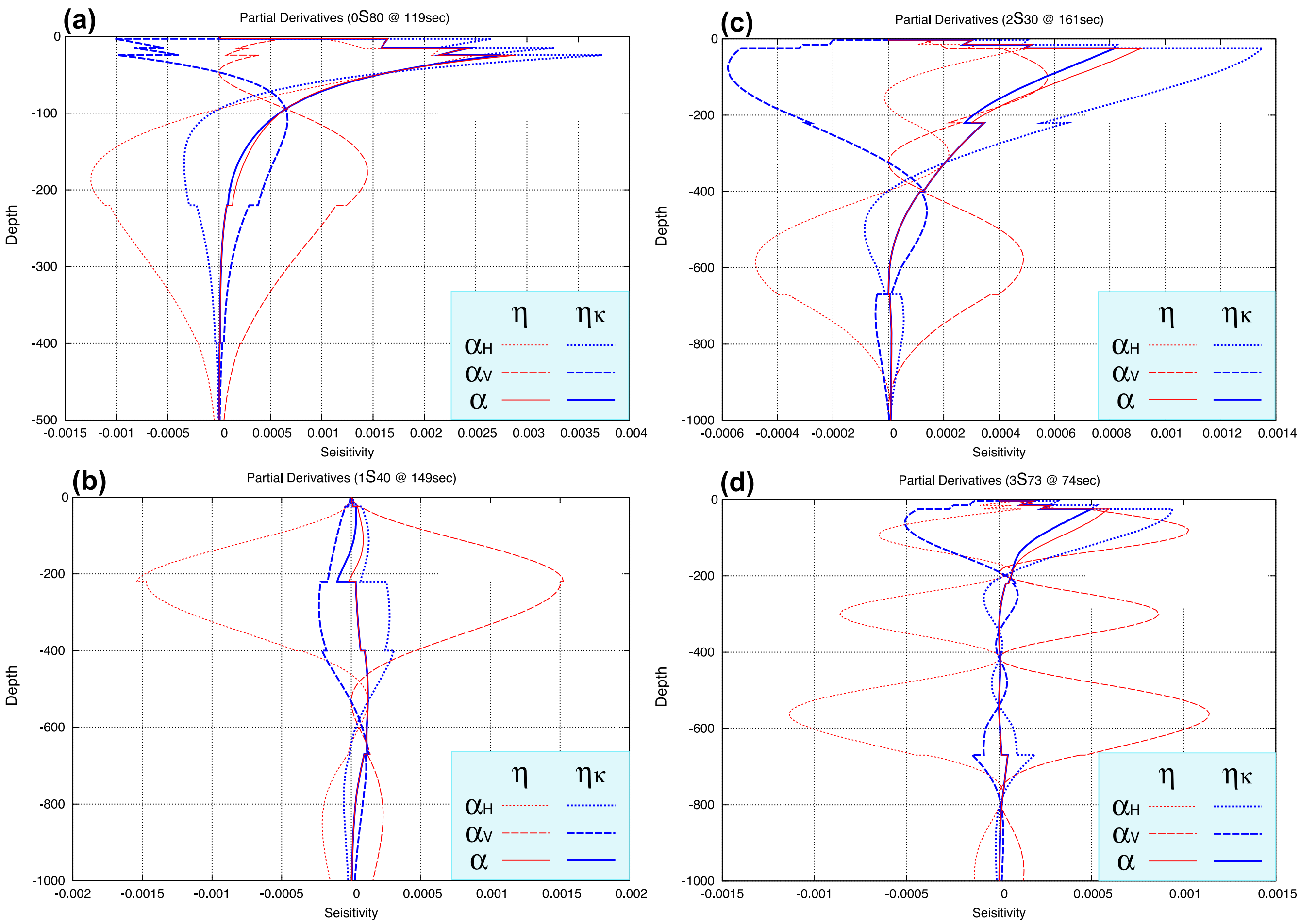

Figure 2. Partial derivatives of spheroidal modes (a) ${}_0S_{80}$, (b) ${}_1S_{40}$, (c) ${}_2S_{30}$, (d) ${}_3S_{73}$ calculated for PREM. Four modes are chosen to be those studied by Anderson & Dziewonski (1982). Blue and red lines denote P-wave partial derivatives for cases with η_κ and η as the fifth parameter. Solid lines are for α_H partial derivatives $\frac{\alpha_H}{\omega}[\frac{\partial\omega}{\partial\alpha_H}]_n^{(*)}$. Note that anisotropic P-wave sensitivity is generally greatly reduced for the η_κ case, but for shallow depths they are comparable or even larger for some modes.

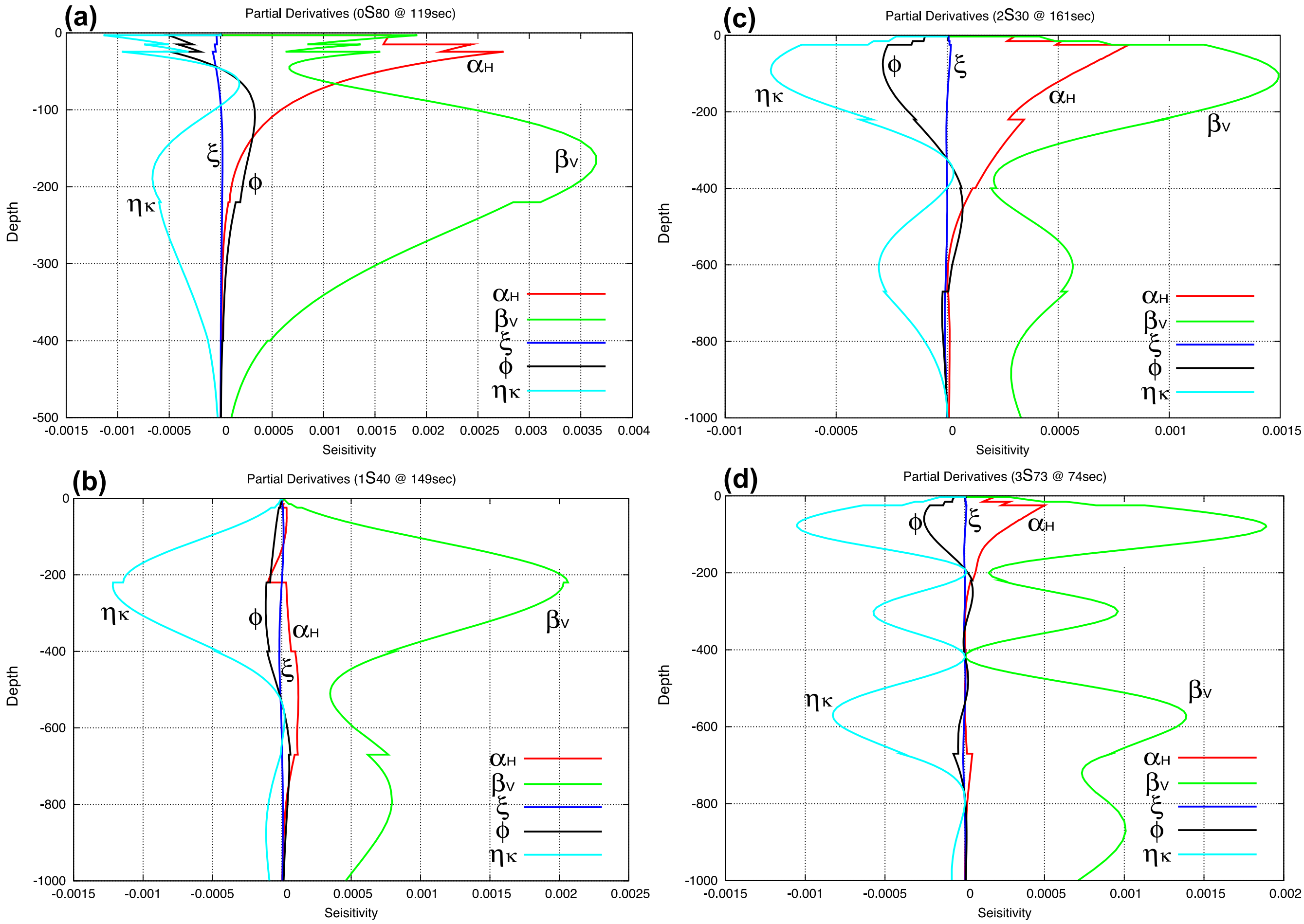

Figure 3. Partial derivatives for the same spheroidal modes (a) ${}_0S_{80}$, (b) ${}_1S_{40}$, (c) ${}_2S_{30}$, (d) ${}_3S_{73}$ as in Fig. 2 for the parameter set (α_H, β_V, ξ, ϕ, η_κ). Red, green, blue, black and cyan lines are for α_H, β_V, ξ, ϕ and η_κ partial derivatives, respectively. Note that the most and the secondmost sensitive parameters are, in general, β_V and η_κ, respectively.

P-wave velocities is generally greatly reduced by the introduction of η_κ. Fig. 3 shows the sensitivity of the same four modes for (α_H, β_V, ξ, ϕ, η_κ) as anisotropic parameters. Again similar to the Rayleigh wave case, the largest and secondmost sensitive parameters are *S*-wave velocity β_V and η_κ. It should be also noted that the shape of sensitivity kernels for the two parameters are similar, as pointed out by Anderson & Dziewonski (1982). Eq. (9) indicates that while η_κ sensitivity $\frac{\eta_\kappa}{\omega}[\frac{\partial\omega}{\partial\eta_\kappa}]_n$ is dictated by the $(r\dot{y}_1\Psi)$ term, β_V sensitivity has an additional large term dependent on $(ry_4)^2$ that is always positive, which make them look alike. This also indicates that good data would be required to resolve them separately (Anderson & Dziewonski 1982).

DISCUSSION

The significant change in the anisotropic *P*-wave parameter sensitivity shown in Fig. 2 may be understood as follows. Fig. 4 demonstrates how phase velocity surfaces of VTI models may change when we modify *P*-wave anisotropy while keeping other parameters constant for the new parametrization using η_κ (left) and the conventional parametrization using η (right). The thick lines show the phase velocity surfaces of *q–P*, *q–SV* and *SH* waves for a reference VTI model that satisfies the elliptic condition (the same reference model shown in fig. 3 of Kawakatsu (2016)). Thin broken lines indicate phase velocity surfaces when we further reduce α_V by changing the elastic constant *C*. To keep η_κ constant (i.e. 1), through eq. (1), *F* needs to be also changed. As the condition η constant does not impose this change in *F*, the ellipticity condition is not satisfied for the case of $\eta =$ const. (a similar situation can occur when *A* or *L* is changed while keeping η constant, as it results in a change in *F* that changes η_κ.) This results not only in the deformation of the *q–P* wave phase velocity surface, but also in the deformation of the *q–SV* wave phase velocity surface. Consequently, an apparent sensitivity of anisotropic *P*-wave parameters occurs due to inappropriately mapped *S*-wave sensitivity, which might have mislead Dziewonski & Anderson (1981) and Anderson & Dziewonski (1982) into their assertion that Rayleigh wave dispersion has such sensitivities. All these seem to follow the line of argument made by Thomsen & Anderson (2015) for the importance of appropriate partial derivatives, although their argument was restricted for weak anisotropy.

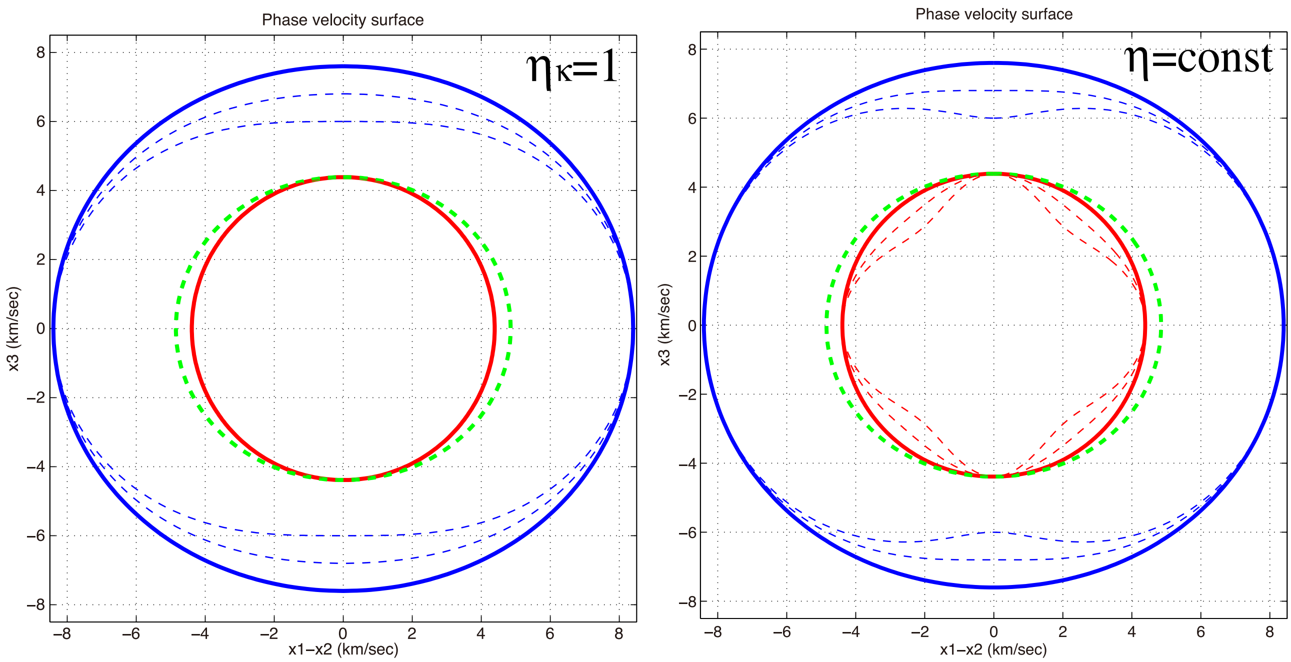

Figure 4. Phase velocity surfaces of q–P (blue), q–SV (red) and SH (green) waves of two sets of VTI models. The thick lines show phase velocity surfaces for a reference VTI model that has 10 per cent anisotropy in both P-wave ($\alpha_{(H,V)}$) and S-wave ($\beta_{(V,H)}$) velocities and satisfies the elliptic condition, $\eta_\kappa = 1$ (the same reference model shown in fig 3 of Kawakatsu (2016)). Thin broken lines show phase velocity surfaces when α_V is further reduced by changing the elastic constant C while keeping other parameters constant for the new parametrization using η_κ (left) and the conventional parametrization using η (right). Note that in the conventional parametrization, by changing α_V while keeping other parameters constant, including $\eta = \text{const}$, we are introducing deformation of phase velocity surfaces, not only for P wave but also for S wave, resulting in apparent sensitivity of anisotropic P-wave parameters.

CONCLUSIONS

Partial derivatives of surface waves and normal modes for new sets of anisotropy parameters of VTI models are investigated. It is shown that incorporation of the new fifth parameter η_κ is beneficial for modeling long-period seismic waves, as well as for body waves.

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APPENDIX: PARTIAL DERIVATIVES FOR THE CONVENTIONAL η

For a comparison purpose, we show partial derivatives for the conventional η given by Takeuchi & Saito (1972) below (some expressions are modified and one typo is corrected):

Rayleigh waves (for the case ($\alpha_H, \alpha_V, \beta_V, \beta_H, \eta$) as independent parameters):

$$\begin{aligned}
\frac{\alpha_H}{c}\left[\frac{\partial c}{\partial \alpha_H}\right]_\omega &= \frac{1}{\omega^2 I_1}\left(\frac{c}{U}\right) kAy_3\left[ky_3 - \frac{2F}{A-2L}\dot{y}_1\right]\\
\frac{\alpha_V}{c}\left[\frac{\partial c}{\partial \alpha_V}\right]_\omega &= \frac{1}{\omega^2 I_1}\left(\frac{c}{U}\right) C\dot{y}_1^2\\
\frac{\beta_V}{c}\left[\frac{\partial c}{\partial \beta_V}\right]_\omega &= \frac{1}{\omega^2 I_1}\left(\frac{c}{U}\right)\left[\frac{1}{L}y_4^2 + \frac{4kFL}{A-2L}\dot{y}_1 y_3\right]\\
\frac{\beta_H}{c}\left[\frac{\partial c}{\partial \beta_H}\right]_\omega &= 0\\
\frac{\eta}{c}\left[\frac{\partial c}{\partial \eta}\right]_\omega &= \frac{-1}{\omega^2 I_1}\left(\frac{c}{U}\right) kF\dot{y}_1 y_3,
\end{aligned} \tag{A1}$$

(for the case ($\alpha_H, \beta_V, \xi, \phi, \eta$) as independent parameters):

$$\begin{aligned}
\frac{\alpha_H}{c}\left[\frac{\partial c}{\partial \alpha_H}\right]_\omega^{(*)} &= \frac{1}{\omega^2 I_1}\left(\frac{c}{U}\right)\left[kAy_3\left(ky_3 - \frac{2F}{A-2L}\dot{y}_1\right) + C\dot{y}_1^2\right]\\
\frac{\beta_V}{c}\left[\frac{\partial c}{\partial \beta_V}\right]_\omega^{(*)} &= \frac{1}{\omega^2 I_1}\left(\frac{c}{U}\right)\left[\frac{1}{L}y_4^2 + \frac{4kFL}{A-2L}\dot{y}_1 y_3\right]\\
\frac{\xi}{c}\left[\frac{\partial c}{\partial \xi}\right]_\omega^{(*)} &= 0\\
\frac{\phi}{c}\left[\frac{\partial c}{\partial \phi}\right]_\omega^{(*)} &= \frac{1}{2\omega^2 I_1}\left(\frac{c}{U}\right) C\dot{y}_1^2\\
\frac{\eta}{c}\left[\frac{\partial c}{\partial \eta}\right]_\omega^{(*)} &= \frac{-1}{\omega^2 I_1}\left(\frac{c}{U}\right) kF\dot{y}_1 y_3,
\end{aligned} \tag{A2}$$

Spheroidal modes (for the case ($\alpha_H, \alpha_V, \beta_V, \beta_H, \eta$) as independent parameters):

$$\begin{aligned}
\frac{\alpha_H}{\omega}\left[\frac{\partial \omega}{\partial \alpha_H}\right]_n &= \frac{1}{\omega^2 I_1}\left[A\Psi\left(\frac{2F}{A-2L}r\dot{y}_1 + \Psi\right)\right]\\
\frac{\alpha_V}{\omega}\left[\frac{\partial \omega}{\partial \alpha_V}\right]_n &= \frac{1}{\omega^2 I_1}C(r\dot{y}_1)^2\\
\frac{\beta_V}{\omega}\left[\frac{\partial \omega}{\partial \beta_V}\right]_n &= \frac{1}{\omega^2 I_1}L\left[\frac{n(n+1)}{L^2}(ry_4)^2 - \frac{4F}{A-2L}r\dot{y}_1\Psi\right]\\
\frac{\beta_H}{\omega}\left[\frac{\partial \omega}{\partial \beta_H}\right]_n &= \frac{1}{\omega^2 I_1}2N\left[n(n+1)y_3(y_1-y_3) - y_1\Psi\right]\\
\frac{\eta}{\omega}\left[\frac{\partial \omega}{\partial \eta}\right]_n &= \frac{1}{\omega^2 I_1}Fr\dot{y}_1\Psi.
\end{aligned} \tag{A3}$$

(for the case ($\alpha_H, \beta_V, \xi, \phi, \eta$) as independent parameters):

$$\begin{aligned}
\frac{\alpha_H}{\omega}\left[\frac{\partial \omega}{\partial \alpha_H}\right]_n^{(*)} &= \frac{1}{\omega^2 I_1}\left[A\Psi\left(\frac{2F}{A-2L}r\dot{y}_1 + \Psi\right) + C(r\dot{y}_1)^2\right]\\
\frac{\beta_V}{\omega}\left[\frac{\partial \omega}{\partial \beta_V}\right]_n^{(*)} &= \frac{1}{\omega^2 I_1}\left[L\left\{\frac{n(n+1)}{L^2}(ry_4)^2 - \frac{4F}{A-2L}r\dot{y}_1\Psi\right\} + 2N\left\{n(n+1)y_3(y_1-y_3) - y_1\Psi\right\}\right]\\
\frac{\xi}{\omega}\left[\frac{\partial \omega}{\partial \xi}\right]_n^{(*)} &= \frac{1}{\omega^2 I_1}N\left[n(n+1)y_3(y_1-y_3) - y_1\Psi\right]\\
\frac{\phi}{\omega}\left[\frac{\partial \omega}{\partial \phi}\right]_n^{(*)} &= \frac{1}{2\omega^2 I_1}C(r\dot{y}_1)^2\\
\frac{\eta}{\omega}\left[\frac{\partial \omega}{\partial \eta}\right]_n^{(*)} &= \frac{1}{\omega^2 I_1}Fr\dot{y}_1\Psi.
\end{aligned} \tag{A4}$$