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A NEW ITERATIVE METHOD FOR FINDING THE NORMAL MODES OF A LATERALLY HETEROGENEOUS BODY

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Abstract. We present a new iterative method for finding the normal modes of a laterally heterogeneous body. The number of operations used by our method is proportional to the square of the number of basis functions used, while the effort required by the variational method increases as the cube of the number of basis functions. A numerical example shows that, at least for lower order modes, our iterative method yields results which are as accurate as the variational method.

Introduction

Since high quality long-period seismograms from the IDA network have become available, the importance of understanding the effect of the earth's lateral heterogeneity on its normal modes has become clear. There are basically two kinds of computational approaches to the lateral heterogeneity problem: perturbation theory and the variational method. Most previous theoretical work on free oscillations of a laterally heterogeneous earth model (summarized by Geller and Stein [1978]) developed solutions based on first order degenerate perturbation theory. This method assumes that the modes of the laterally heterogeneous body are linear combinations of only the singlets in the original degenerate multiplet, with no contributions from any adjacent multiplets. However Geller and Stein [1978] and Morris and Geller [1981] showed that perturbation theory, in spite of its computational simplicity, seemed insufficiently accurate for applicability to the earth. It is important to note that the computational objective is not merely to find the eigenfunctions and eigenfrequencies of the laterally heterogeneous earth model "to first order," but rather to obtain an accurate estimate of the *difference* between the modes of the laterally heterogeneous model, and the average, spherically symmetric, model.

The variational (Rayleigh-Ritz) method proposed by Geller and Stein [1978] gives sufficiently accurate results. However, as mentioned by Morris and Geller [1981], who extended the variational method to a simple laterally heterogeneous sphere, formidable computational difficulties exist in applying the variational method to a realistic earth model, because the solution of a very large matrix eigenvalue problem is required.

In this paper, we present a new iterative method, which for many cases appears to combine the accuracy of the variational technique with the computational speed of perturbation methods. We present a numerical example demonstrating our method's applicability to lower order modes.

Second Order Perturbation Theory

Following Geller and Stein, the lateral heterogeneity problem in which we are interested can be written as

$$H|\psi\rangle + \omega^2 \rho\,|\psi\rangle = 0 \qquad (1)$$

and the boundary conditions as

$$\alpha|\psi\rangle + \beta\frac{\partial|\psi\rangle}{\partial n} = 0 \qquad (2)$$

where H and ρ are hermitian operators for potential and kinetic energy respectively, ω is the eigenfrequency, $|\psi\rangle$ is the eigenfunction and α and β are arbitrary functions of position on the boundary.

We suppose that the unperturbed eigenvalue problem,

$$H^{(0)}|\psi^{(0)}\rangle + \rho^{(0)}(\omega^{(0)})^2|\psi^{(0)}\rangle = 0 \qquad (3)$$

has already been solved. For the earth, this means that the eigenfrequencies and eigenfunctions of the corresponding spherical, vertically heterogeneous model are known.

When a small change in the operators, $H^{(1)}$, and $\rho^{(1)}$, is introduced into the system, the eigensolutions become

$$\omega^2 = (\omega^{(0)})^2 + (\delta\omega)^2, \quad |\psi\rangle = |\psi^{(0)}\rangle + |\delta\psi\rangle \qquad (4)$$

where $(\delta\omega)^2$ and $|\delta\psi\rangle$ are the corresponding change in the eigenvalue and eigenfunction respectively. As is commonly done in quantum mechanics [e.g., Schiff, 1968], we expand the perturbed eigenfunction in terms of the unperturbed modes,

$$|\psi_n\rangle = \sum_m C_{nm}\,|m\rangle, \qquad (5)$$

where $|m\rangle \equiv |\psi_m^{(0)}\rangle$ and C_{nm} is the expansion coefficient of the original m*th* mode in the expansion of the perturbed n*th* mode. In the rest of this paper, n and N will denote the order number of the mode whose solution we are seeking and the number of the trial functions used in the expansion, respectively. Our problem is to find C_{nm}'s for each of the eigenfunctions of the laterally heterogeneous model.

According to Geller and Stein [1978], second-order perturbation theory gives

$$C_{nm} = \frac{\langle m|H^{(1)}|n\rangle + (\omega_n^{(0)})^2\langle m|\rho^{(1)}|n\rangle}{(\omega_m^{(0)})^2 - (\omega_n^{(0)})^2}, \quad m \neq n \qquad (6)$$

$$C_{nn} = 1 \qquad (7)$$

Once the matrix elements $\langle m|H^{(1)}|n\rangle$ and $\langle m|\rho^{(1)}|n\rangle$ are obtained (these matrix elements are also required for the variational method), the number of operations required to find C_{nm} is proportional to the number of trial functions, N.

Variational Method

In the variational method, we substitute (5) into the Lagrangian and search for C_{nm}'s which satisfy Hamilton's Principle, which requires that the Lagrangian, L, must be stationary for any arbitrary small

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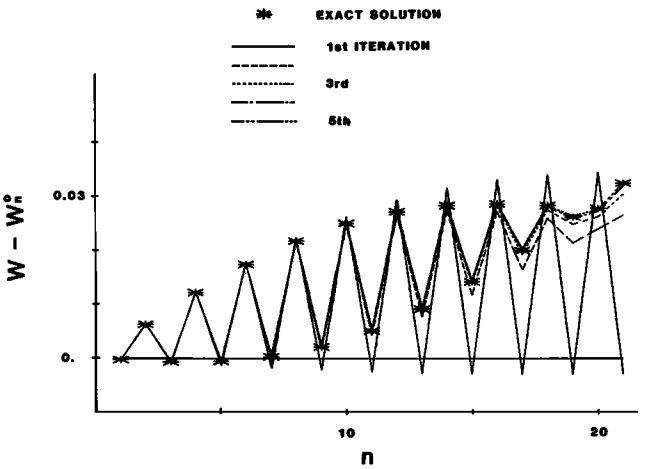

Figure 1. Successive iterations of the new method for modes n=1 to 21. Note that after four iterations, there is excellent agreement between the eigenfrequencies from the exact solution (stars) and the results of the iterative method.

change in the C_{nm}'s. Differentiating L with respect to C_{nm}, we obtain a matrix eigenvalue problem

$$V\ \mathbf{C}_n = \omega_n^2\ T\ \mathbf{C}_n \tag{8}$$

where the components of $\mathbf{C}_n$ are C_{nm}, and T and V are given by

$$T_{ij} = \langle i|\rho|j\rangle = \langle i|\rho^{(1)}|j\rangle + \delta_{ij} \tag{9}$$

and

$$V_{ij} = -\langle i|H|j\rangle = -\langle i|H^{(1)}|j\rangle + \delta_{ij}(\omega_j^{(0)})^2 \tag{10}$$

Geller and Stein used inverse iteration to find both the eigenfrequencies and eigenvectors of (8) in their test case. While other methods may be required if a larger number of trial functions is used, any variational method will require (essentially) the solution of N simultaneous equations, and the number of operations required for a direct solution will be proportional to N^3. This represents a very serious problem in applying the variational method to realistic earth models, because N will be on the order of several thousand.

New Iterative Method

In (8), if we knew the eigenvalue, ω_n^2, we could find the eigenvector, $\mathbf{C}_n$, by solving the following $N-1$ simultaneous equations.

$$A\ \mathbf{x} = -\mathbf{b} \tag{11}$$

where the $N-1 \times N-1$ matrix, A, is formed by omitting the n*th* row and the n*th* column of the $N \times N$ matrix, $(V - \omega_n^2 T)$, and the $N-1$ dimensional vectors $\mathbf{x}$ and $\mathbf{b}$ are $\mathbf{C}_n$ and the n*th* column of $V - \omega_n^2 T$ without the n*th* element, respectively. The eigenvector is given by

$$\mathbf{C}_{nm} = \begin{cases} x_m & m<n \\ 1 & m=n. \\ x_{m-1} & m>n \end{cases} \tag{12}$$

In order to solve (11), a variation of the Jacobi iteration method is used.

$$D\ \mathbf{x}^{(k+1)} = -F\ \mathbf{x}^{(k)} - \mathbf{b} \tag{13}$$

where the matrices D and F are the diagonal and off-diagonal part of A. Because the exact eigenfrequency is not known, we use $\omega_n^{(0)}$ as the initial guess. For the initial guess of the eigenvector, we use the unperturbed mode, $C_{nm}^{(0)} = \delta_{nm}$. After each iteration, the Rayleigh quotient is used to estimate the new eigenfrequencies, i.e.,

$$(\omega_n^{(k)})^2 = \frac{\mathbf{C}_n^T\ V\ \mathbf{C}_n}{\mathbf{C}_n^T\ T\ \mathbf{C}_n}. \tag{14}$$

Because of the choice of the initial guess, the first iteration yields exactly the same eigenvector as second-order perturbation theory. After several iterations have resulted in convergence, we normalize the eigenfunction,

$$|\psi_n\rangle = \frac{\sum_m C_{nm}\ |m\rangle}{\sqrt{R}} \tag{15}$$

where

$$R = \mathbf{C}_n^T\ T\ \mathbf{C}_n \tag{16}$$

The number of operations required by our iterative method is proportional to N^2.

Numerical Example

We apply our new method to exactly the same problem considered by Geller and Stein[1978]: a one-dimensional inhomogeneous string with a jump in the stiffness and the density at the center. We use N=41 (41 trial functions, centered around the n*th* unperturbed mode), so we can obtain a direct comparison with Geller and Stein's variational test. Our results are shown in Figure 1, 2 and 3.

In Figures 1 and 2, we show results for modes from n=1 to n=21. The convergence of the iterative method is extremely good. Figure 1 shows that four iterations are sufficient to obtain excellent convergence of the frequencies to the exact solution for all of the modes up to n=21. The expansion coefficients for each per-

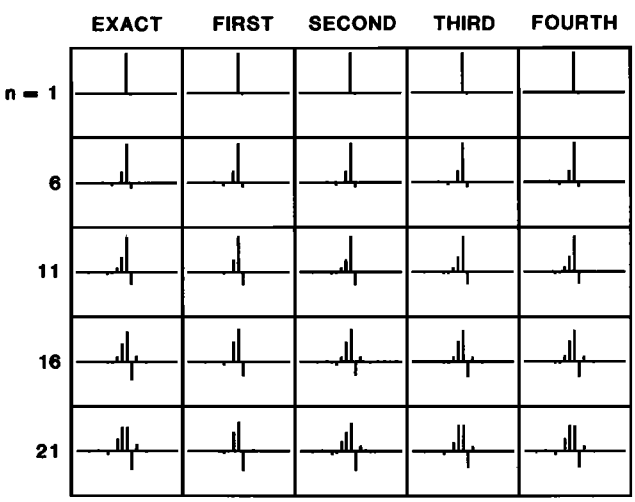

Figure 2. Successive expansion coefficients for modes n=1 to 21. The center "stick" in each box gives the coefficient of the original eigenfunction in the expansion of the perturbed mode. Sticks to the left and right indicate coefficients of original eigenfunctions successively higher and lower respectively. The coefficients of modes an even distance from the original mode (*forbidden* modes) are zero after the first iteration, but converge to their correct values after further iterations.

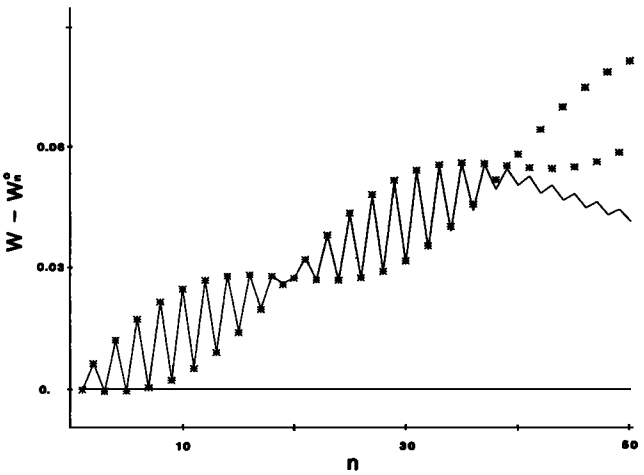

Figure 3. Eigenfrequencies from the iterative method (after convergence) compared to the exact solution (stars). Agreement is excellent until about $n=40$.

turbed mode (in terms of the unperturbed modes) are plotted in Figure 2. Note that in the first iteration the coefficients for the modes which are an even distance away from the "parent mode," are zero because of the selection rules for the matrix elements. In later iterations, however, these *forbidden* modes have nonzero coefficients.

Figure 3, in which we extend the calculation up to $n=50$, shows the discrepancy between the exact solutions and the approximations from above about $n=40$. This is because the perturbation is so large that the Rayleigh quotient no longer yields a good approximation of the eigenvalue. In this case, the iterative method does not converge to the correct solution. However, Figure 2 of Geller and Stein [1978] shows that for $n=31$ or $n=41$, the expansion coefficient of the "parent mode" in the exact solution is much smaller (and is almost zero for $n=41$) than the coefficients of neighboring modes. In other words, the iterative method appears not to succeed in cases where it no longer is appropriate to think of the perturbation to the laterally homogeneous problem as "small".

Discussion

When we are finding even the lowest modes of a realistic laterally heterogeneous earth model, the number of trial functions will rapidly become extremely large. For example, if we want to find the split eigenfrequencies and eigenfunctions of ${}_0S_{10}$ (21 split singlets) we might reasonably use a basis set, $|n>$ consisting of both spheroidal and torsional modes, $|k\ l\ m>$, where k, l and m are overtone, angular order and azimuthal order number, respectively. If we use all unperturbed singlets with $k=0$ to 4, $l=0$ to 21 and $m=-l$ to l, we will have a basis set, $|n>$ with about $N\approx4400$ modes. If the iterative method proposed in this paper proves applicable to this case, the reduction in the number of computations clearly will be enormous.

When the exact eigenvalue is not known, i.e., $\det(V-\omega^2 T) \neq 0$, solving (11) and (12) gives a solution, $\mathbf{C}_n$, for which

$$(V - \omega_n^2 T)\,\mathbf{C}_n = (0,\ldots,0,\delta_n,0,\ldots,0)^t \qquad (17)$$

where

$$\delta_n = \frac{\det(V - \omega_n^2 T)}{\det(A)}. \qquad (18)$$

When our iterative method converges to a solution, we can test for its correctness by checking δ_n. (Because convergence was obtained, the other elements will be equal to zero.) Thus, δ_n will be a very good criterion for testing a solution obtained using the iterative method.

To apply our method to a problem for which the original unperturbed eigenvalue problem has degenerate multiplets, first order perturbation theory must first be used to split the multiplets into non-degenerate singlets. Kawakatsu [1981] and Morris *et al.* [1981] successfully applied this technique to a heterogeneous square membrane and a simple laterally heterogeneous sphere. Their results also suggest the efficacy of the iterative method.

The success of the present iterative method suggests that there might exist another iterative method, which is uniformly convergent, in solving such diagonally dominant matrix eigenvalue problems. We are currently working on this problem.

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