Variability of the topographic core-mantle torque calculated from core surface flow models

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Abstract

With the prospect of studying the relevance of the topographic core-mantle coupling to the variations of the Earth’s rotation and also its applicability to constraining the core surface flow, we investigate the variability of the topographic torque estimated by using core surface flow models accompanied by (a) uncertainty due to the non-uniqueness problem in the flow inversion, and (b) variance originating in that of geomagnetic secular variation models employed in the inversion. Various flow models and their variances are estimated by inverting prescribed geomagnetic models at the epoch 1980. The subsequent topographic torque is then calculated by using a core-mantle boundary topography model obtained by seismic tomography. The calculated axial and equatorial torques are found subject to the variability of order $10^{19}$ and $10^{20}$ Nm, respectively, on which (b) is more effective than (a). The variability of the torque is attributed even to (a) and (b) of the large-scale flows (degrees 2 and 3). Yet, it still seems unlikely for the decadal polar motion with the observed amplitude to be excited exclusively by the equatorial topographic torque associated with any of reasonable core surface flow models. It is also confirmed that, with the topography model adopted here, the axial topographic torque on a rigid annulus in the core (coaxial with the Earth’s rotation axis) associated with any of reasonable flow models is larger by two orders of magnitude than the plausible inertial torque on such cylinders. This implies that any core surface flow model consistent with the topographic coupling does not exist, unless the topography model is appropriately modified. Nevertheless, the topographic coupling might provide not only a weak constraint for explaining the decadal LOD variations, but also the possibility to probe the core surface flow and the core dynamics.

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1. Introduction

Historical observations of the Earth’s rotation have revealed that the Earth’s angular velocity fluctuates on various timescales. For both the length of day (LOD) variations and the polar motions, decadal variations have been confirmed to comprise one of their predominant components (Rochester, 1984). The decadal variations of the LOD appears well correlated with that of the geomagnetic field (e.g. Le Mouël et al., 1981), so it has long been envisaged that the Earth’s rotation has some connection to the fluid motion at the core surface which is considered responsible for the decadal variations of the geomagnetic field.

Currently, the decadal LOD variations are attributed more certainly to the dynamical interaction between the core and mantle. Its decisiveness is provided by the similarity of the observed and predicted LOD variations on decade timescales, the latter of which is deduced based on a time series of core surface flow model estimated from geomagnetic data (e.g. Jackson et al., 1993). According to the observed LOD fluctuations (several milliseconds in decades), the axial torque that should be
exerted on the mantle is estimated to be $10^{18}$ Nm at maximum. But the mechanism to dynamically couple the core and mantle is still poorly known, while there are some plausible candidates: the electromagnetic, gravitational and topographic core-mantle coupling (e.g. Ponsar et al., 2003). It is noteworthy that a particular type of the electromagnetic core-mantle torque associated with induced currents in the conducting mantle due to the advection of radial magnetic field at the core surface (or the frictional part of the electromagnetic torque) alone may suffice to explain time series of the decadal LOD variations without being inconsistent with geomagnetic data (Holme, 2003). It is noteworthy that a particular type of the electromagnetic core-mantle coupling could be the possible mechanism for the core-mantle coupling, though whether it plays the dominant role is still inconclusive. As for the gravitational and topographic coupling, no model has been obtained yet for explicitly explaining the observed LOD variations.

The decadal polar motion, or the Markowitz wobble, has been detected as the wander of the mean pole superimposed on its secular drift, identified by filtering out its motion on shorter timescales such as the Chandler and annual wobbles (e.g. Dickman, 1981). The Markowitz wobble is retrograde and highly elliptic (with the amplitude of several tens of milliarcseconds) and its period is approximately 30 years (Poma, 2000). Since this period is out of those of the Earth’s free wobble, it should be excited by external torque on the mantle, of which magnitude is expected to be of the order of $10^{20}$ Nm. Unlike the case of the axial torque of the same timescales, no consensus has been reached as to which part of the Earth exerts the equatorial torque on the mantle. The topographic core-mantle coupling is thought to be one of the possible exciting mechanisms of the Markowitz wobble (Hinderer et al., 1987).

The focus of the present paper is the topographic core-mantle coupling, first proposed by Hide (1969). The topographic torque on the mantle is expected to arise from the net effect of fluid pressure of the outer core acting on the bottom of the mantle which has aspherical topography (Section 2.1). To study this torque two approaches have been developed, i.e. those based on observations (e.g. Hide, 1986; Jault and Le Mouël, 1989) and numerical modeling (e.g. Kuang and Bloxham, 1993; Kuang and Chao, 2003). Our present work goes along the former.

As an observational approach to investigate the topographic coupling, the method to calculate the topographic torque was advocated by Hide (1986) (we call it “Hide’s method” hereafter). In Hide’s method, models of the core-mantle boundary (CMB) topography (provided by seismic tomography) and the pressure field at the core surface are employed. The pressure is derived from a core surface flow model which is obtained by inverting geomagnetic main field (MF) and its secular variation (SV). To relate the flow and pressure, a priori constraint of “tangential geostrophy (TG)” has to be imposed on the flow, where the Coriolis force is equilibrated by the pressure gradient force (and irrotational component of the buoyancy and Lorentz forces) in the horizontal force balance at the CMB. The TG hypothesis may be verified by virtue of the successful result of predicting the decadal LOD variations by applying TG flows (see Pais et al., 2004) (except in the non-geostrophic belt (Chulliat and Hulot, 2000), where the influence of the Coriolis force does not substantially exceed that of the Lorentz force). The calculation of the axial topographic torque by Hide’s method was first implemented by Speith et al. (1986). They showed that the calculated torque is larger by more than one order of magnitude than what might account for the observed decadal LOD variations. This result was also confirmed by Jault and Le Mouël (1989) (they attached to Hide’s method more detailed reasoning by allowing for the role of inertial force on the outer core fluid to complete the angular momentum budget in the core and mantle (see Section 2.1.1)). The calculation of the equatorial topographic torque by Hide’s method has also been attempted, which indicates that its typical magnitude is smaller by several factors than that corresponding to the Markowitz wobble (Greff-Lefftz and Legros, 1995; Hide et al., 1996; Hulot et al., 1996). The time series of these calculations achieved no significant correlation in phase with the observation. In both axial and equatorial cases, simple application of Hide’s method fails to predict the variation of Earth’s rotation.

It is worth regarding, however, that there is a remarkable feature with the torques calculated by Hide’s method: they are extremely sensitive to adopted models of the flow and topography which are known to be anything but robust (e.g. Jault and Le Mouël, 1990; Kuang and Bloxham, 1993). As a matter of fact, even a slight relative shift of these two models can lead to a large variation in the consequential torque (Jault and Le Mouël, 1990). This is because the net torque results from cancellation of local forcing which has the potential to generate the torque as large as $10^{20}$ Nm in total, which is readily found by an order of estimate (Roberts, 1988). This fact may imply that the approach to examine the resemblance between predicted and observed LOD variations by simply applying Hide’s method is of little relevance for investigating the hypothesis of topographic core-mantle coupling as the influential mechanism.

To address whether Hide’s method and the postulated coupling mechanism are viable, it may be necessary to
make sure if there is the core surface flow capable of explaining the time series of the variation of the Earth’s rotation, as has been shown in the case of the electromagnetic coupling (Holme, 1998). For that to be realized, the calculated axial topographic torque must at least be reduced so as to get reconciled with the LOD observations, whereas the equatorial torque may fail to reach the observational magnitude for the Markowitz wobble, yielding the dominant role to other mechanisms such as associated with torques at the inner core boundary (Dumberry and Bloxham, 2002). Such reduction would take place even in the context of Hide’s method, provided that the primary core flow is strongly influenced by the CMB topography to which the flow is locked due to some stabilizing mechanisms (Jault and Le Mouël, 1999, 2003). This primary flow appears to correspond mainly to the convection within the core or thermal wind associated with the lower mantle’s lateral temperature variation (Gibbons and Gubbins, 2000). We note that, if the primary flow has latitudinal component at the core surface (except along the equator) unlike the analyses of Kuang and Bloxham (1993) or Braginsky (1999) in which they assume purely zonal flow for it, a local topographic torque arises that can potentially lead to excessive torque as a whole (Appendix A.1). So the true primary flow might have a specific distribution that allows nearly complete cancellation in the net torque. Indeed, the dynamical study of the topographic torque with full dynamo computation implies that the core flow quickly evolves in accordance with the topography so that it may give rise to the topographic torque of marginal significance over a long timescale (Kuang and Chao, 2003).

The advantage of the axial topographic torque as a result of cancellation of potentially large forcings is that it can readily explain the LOD variations on timescales even shorter than those of the SV and the core surface flow; what is more, further constraints on the flow might be provided by the requirement of reducing the magnitude of the axial torque. Therefore, with the prospect of examining the dynamical consistency of the topographic coupling and simultaneously enhancing the reliability of flow models, it is worth probing systematically the flow models. Indeed, the dynamical study of the topographic torque by Hide’s method (see Section 2.1.2) is expected to be weakly constrained by geomagnetic data, for they seem to be very well explained by toroidal component of the flow alone (e.g. Whaler, 1986). Failure of given assumptions can, of course, lead to additional ambiguity (this would be, in the case of Hide’s method, incomplete adaptability of the FF and TG assumptions on account of joint effect of the magnetic diffusion and the Lorentz force at the CMB (Hulot and Chulliat, 2003)). Second, we have limited observational resolution of the geomagnetic field within the Earth; geomagnetic models are always truncated, and necessarily, so are flow models. The MF above degree 13 is known to be dominated by the crustal field, and the observed SV above degree 8 may have little concern with the true core signal. Hulot et al. (1992) argue that geomagnetic data are suspected to impose no constraint on the core surface flow above degree 8. Moreover, the SV below degree 8 may be generated also by the flow and/or the MF above their truncation degrees (the non-modeled SV, Eymin and Hulot, 2005), and hence truncated models of the flow can be contaminated by the aliasing due to the underparameterization. But it is worth keeping in mind that the true flow might be of large-scale (i.e. its spectrum converges below as low degree as 8), which is implied by the alternative approach where the geostrophic pressure at the core surface (Section 2.1.1) is computed without relying on smoothing assumptions (Chulliat and Hulot, 2000, they refer to it as the local method in contrast with the spectral method, that is, the conventional spherical harmonic approach). If this is the case, an appropriate damping will serve to provide us with a good estimate of the flow (Celaya and Wahr, 1996). Third, estimated flows are accompanied by variance originating in that of geomagnetic models. In particular, SV models will be afflicted by considerable variance even on large spatial scales, because of poor coverage of the magnetic stations at the Earth’s surface whose SV data contribute to no small extent to the models (Barker and Barraclough, 1985). Further variance can be generated through their modeling process; constructed SV models at a certain time will differ from each other (even if satellites’ abundant data were available) depending on their parameterization, a priori beliefs about them, and selection or weighting of data. Downward continued field at the
CMB should be associated with further ambiguity due to the conductivity of the mantle, but this appears to be relatively smaller than that originating in variance of a geomagnetic model itself.

In the present study, we investigate the range of topographic torque associated with the uncertainty of the core surface flow. We limit our analysis to the flows that exactly obey the assumptions of FF and TG, not only because they are necessary for Hide’s method but also because they are thought likely to well reflect the true flow to the first approximation and adopted in many of the recent papers (e.g. Gire and Le Mouël, 1990; Jackson et al., 1993; Jackson, 1997; Puis and Hulot, 2000). Three components of the torque and their variances are studied by using flow models available for a particular epoch 1980. For the CMB topography, a model obtained recently by Boschi and Dziewonski (2000) is prescribed as a reference model. This model provides much higher resolution and somewhat larger amplitude than those which are used in the earlier computations of the topographic torque.

In Section 2, we give an overview of the topographic torque and Hide’s method, and describe the scheme of the inversion for obtaining various TG flows. The mod- ular purpose; we examine in the first part the variance of the SV model. We then proceed to the analysis for our purpose; we examine in the first part the variance of the SV model. We then proceed to the analysis for our

2. Theory and method

4. The conclusions are presented in Section 6.

2.1. The topographic core-mantle torque

The topographic torque on the mantle \( \mathbf{\Gamma} \) exerted by the fluid pressure \( p^* \) of the outer core is defined by surface integral over the bumpy CMB, or equivalently volume integral over the whole core \( V_C \):

\[
\mathbf{\Gamma} \equiv \int_{\text{CMB}} \mathbf{r} \times p^* \mathbf{n} \cdot dS = \int_{V_C} \mathbf{r} \times \nabla p^* \, dV
\]

(1)

where \( \mathbf{r} \) is a position vector with respect to the Earth’s center and \( \mathbf{n} \) is an outward unit normal vector at the CMB. \( \mathbf{\Gamma} \) (\( \mathbf{\Gamma}_x, \mathbf{\Gamma}_y, \mathbf{\Gamma}_z \)) is based on Cartesian coordinates fixed to the uniformly rotating mantle. The axis that penetrating geographic poles is defined as \( \zeta \). Other two axes, \( x \) and \( y \), are those within the equatorial plane crossing 0 and 90°E meridians, respectively. \( \mathbf{\Gamma}_e \) is referred to as the axial torque, while \( \mathbf{\Gamma}_x \) and \( \mathbf{\Gamma}_y \) are referred to as the torques of geophysical relevance. In this subsection, we outline Hide’s method to follow its essence explicated by Jackson et al., 1993; Jackson, 1997; Pais and Hulot, 2000. More specifically, if the topographic coupling is supposed to be the only mechanism for exciting the decadal variations of the Earth’s rotation, the reaction topographic torque on the core \( \mathbf{\Gamma} \) should be equilibrated kinematically by the moments of inertial and Coriolis forces on the core (\( \mathbf{\Gamma}_I \) and \( \mathbf{\Gamma}_C \), respectively). \( \mathbf{\Gamma}_I + \mathbf{\Gamma}_C = -\mathbf{\Gamma} \). Their expressions involve the flow field \( u^* \) in the core with respect to the rotating mantle with its angular velocity \( \mathbf{\Omega} \) (\( \equiv \mathbf{\Omega}(t) + \mathbf{\Omega}_0 \equiv (\omega_x, \omega_y, \omega_z + \Omega_0) \)):

\[
\mathbf{\Gamma}_I \left( \frac{\partial u^*}{\partial t} \right) = \int_{V_C} \mathbf{r} \times \rho_0 \left( \frac{\partial u^*}{\partial t} + \frac{\partial \mathbf{u}^*}{\partial t} \right) \, dV
\]

\[
\simeq \int_{V_C} \mathbf{r} \times \rho_0 \frac{\partial u^*}{\partial t} \, dV
\]

(2)

where \( \rho_0 \) is the mean core density (with the Boussinesq approximation adapted), and

\[
\mathbf{\Gamma}_C(u^*) = \int_{V_C} \mathbf{r} \times (2\mathbf{\Omega}_0 \mathbf{\Omega} \cdot \mathbf{u}^*) \, dV.
\]

(3)

More specifically, \( \mathbf{\Gamma}_C \) can be balanced only by \( \mathbf{\Gamma}_C^z \) because \( \mathbf{\Gamma}_C^x \) is equal to zero

\[
\mathbf{\Gamma}_C^z(u^*) = 0.
\]

(4)

which is derivable by using fluid incompressibility \( \nabla \cdot u^* = 0 \) and impermeable condition \( \mathbf{n} \cdot u^* = 0 \) at the CMB (e.g. Jault and Le Mouël, 1989). On the contrary, \( \Gamma_{x,y} \) is in balance mostly with non-zero \( \Gamma_{x,y}^C \) on decade timescales, implying that the decadal polar motion can be considered as manifestation of quasi-static equilibrium between them (\( \Gamma_{x,y}^I \) and \( \Gamma_{x,y}^C \) are balanced on much shorter timescales to excite the free nutations).

2.1.1. Hide’s method

The principle of Hide’s method lies in the estination of \( \Gamma \) by applying models of the CMB topography and the pressure anomaly acquired respectively from seismic and magnetic observations. It seems that the theory of the method is somewhat intricate and still divided into whether or not its outcome reflects the topographic torque of geophysical relevance. In this subsection, we outline Hide’s method to follow its essence explicated by Jault and Le Mouël (1989).

Seismic tomography provides a model of the devi- ation of the CMB from the axisymmetric hydrostatic
ellipsoid of the outer core figure. The CMB topography \( r_{\text{CMB}}(\theta, \phi) \) can be represented by sum of the mean core radius \( \epsilon = 3485 \text{ km} \) and the deviating topography \( h^*(\theta, \phi) \)

\[
r_{\text{CMB}} = \epsilon + h^*(\theta, \phi) = \epsilon(1 + \eta h(\theta, \phi))
\]

in spherical polar coordinates \((r, \theta, \phi)\), where \( h(\sim O(1)) \) is the non-dimensional CMB topography and the small parameter \( \epsilon \) is the ratio of typical amplitude of the CMB topography to the mean core radius \( h^* = \epsilon h(c) \), or comparable, to horizontal scale \( L_0(c) \) of the topography \( (c \sim O(10^{-7})) \).

The pressure anomaly at the CMB, \( p^*(r_{\text{CMB}}) \), may be associated with the fluid flow \( u^*(r_{\text{CMB}}) \), which complies with the impermeable condition \( \hat{n} \cdot u^*(r_{\text{CMB}}) = 0 \). Because the typical amplitude of the topography is much smaller than \( \epsilon \) or \( L_0 \) (i.e. \( \epsilon \ll 1 \)), \( u^*(r_{\text{CMB}}) \) may be approximated by \( u^*_0(c) \), a model of horizontal flow over the spherical surface at \( r = \epsilon \) which can be estimated by inverting geomagnetic data (subscript \( H \) denotes the horizontal component, e.g. \( u_H = u - \mathbf{r} \times \mathbf{u}_r, \mathbf{V}_0 = \nabla - \mathbf{F}_0, \mathbf{r} \) being the unit radial vector). Indeed, \( u^*_0(c) \) can be considered as belonging to the primary core flow \( u^*_0 \) associated with convection within the spherical core. The total core flow \( u^{*} \) is given by the sum of \( u^*_0 \) and perturbation flow due to the boundary topography \( h^* \), which may be represented by the expansion of \( u^{*} \) in powers of \( \epsilon \) (Jault and Le Mouël, 1999):

\[
u^*_0 = u^*_0 + u^*_1 + \cdots = U(u_0 + cu_1 + \cdots),
\]

where \( u_0, u_1(\sim O(1)) \) are non-dimensional flows and \( U \) is typical flow velocity \((U \sim 5 \times 10^{-3} \text{ m s}^{-1})\).

Then, TG (Eq. (10)) for \( p^*(r_{\text{CMB}}) \) and \( u^*_0(c) \) also holds at the zeroth order in \( \epsilon \). Accordingly, \( p^*(r_{\text{CMB}}) \) can be obtained from a TG flow model \( u^*_0(c) \) within the error of order \( \epsilon \).

Note here that \( p^*_0(c) \) obtained from \( u^*_0(c) \) by Eq. (10) does not necessarily coincide with the fluid pressure \( p^*_0 \) but contains other potentials \( \Pi \) such as those of centrifugal or radially irrotational Lorentz and buoyancy forces (Jault and Le Mouël, 1989; Hulot et al., 1996). Nevertheless, these extra pressures will not in theory affect the principle of Hide’s method. Hereafter, we call \( p^*_0(c) \) as the geostrophic pressure.

Now, substituting Eqs. (5) and (9) into Eq. (1) and converting it into a spherical surface integral (e.g. Roberts, 1988), one will obtain

\[
\Gamma = \Gamma_1 + \Gamma_2 + \cdots = 4\pi \epsilon^3 \rho \times \left( \frac{\epsilon}{r} \int_0^{2\pi} \int_0^\pi \mathbf{h} \cdot \nabla \mathbf{u}_0 \sin \theta d\theta d\phi + O(\epsilon^2) \right)
\]

where \( \mathbf{V}_0 = \mathbf{r} \mathbf{u}_0 \) is non-dimensional horizontal gradient. The first term on the rhs of the above equation, \( \Gamma_1 \), is the estimate of the topographic torque by Hide’s method, given as an approximation of \( \Gamma \) to within second-order term in \( \epsilon \).

It is to be noticed that, if \( u^*_0 \) satisfies Eq. (10), then

\[
-\Gamma_1 \sim \Gamma_2^0(u_0^0), \text{ i.e. the axial component of the topographic pressure on the core is identical to the axial Coriolis torque calculated only from the primary flow } u^*_0 \text{ (the fictitious Coriolis torque, Jault and Le Mouël, 1989).}
\]

In fact, unlike Eq. (4), \( \Gamma_2^0(u_0^0) \) does not in general vanish because \( u^*_0(r_{\text{CMB}}) \) is not necessarily subject to the impermeable condition \((\hat{n} \cdot u^*_0(r_{\text{CMB}}) = 0)\); it is actually dependent on the product of the topography \( h^* \) and the latitudinal component of the primary flow \( u^*_0 \) at \( r = \epsilon \) (see Appendix A.1). It follows that, except for the specific case in which the distribution of \( u^*_0(r_{\text{CMB}}) \) is correlated with the CMB topography, the leading term of \( \Gamma_2 \) is of the first order in \( \epsilon \). Furthermore, it is shown at length by Jault and Le Mouël (1989) that the perturbation flow \( u^*_0 \) governed by

\[
\frac{\partial u^*_0}{\partial t} = \nabla^2 u^*_0 + \mathbf{F}_0 \cdot u^*_0 + \mathbf{F}_0 \times (\mathbf{u}_r \times \mathbf{u}_L), \text{ i.e. } F_{\text{H}}(c) = \nabla \times \mathbf{F}_0 = 0,
\]

in the first order force balance in \( \epsilon \), the inertial term is included which has been safely dropped in Eq. (7) because of the Rossby number not being larger than \( \epsilon \) (Jault and Le Mouël, 1999) and by the boundary condition

\[
\hat{r} \cdot u^*_0(c) = -\hat{n} \cdot u^*_0(r_{\text{CMB}}) \equiv g(r_{\text{CMB}}, \hat{r}, \phi)
\]
gives rise to the Coriolis torque \( \Gamma_f \) which proves to be balanced by the axial angular acceleration of the core (i.e. \( \Gamma_f = 0 \)). Since \( \Gamma_f^\alpha(\mathbf{n}) = 0 \) (Eq. (4)), one can find the axial angular momentum equation for the core \( \Gamma_{CMB} = \frac{1}{2} \hat{\mathbf{n}} \) as summarized in Jault and Le Mouël (1999). It is thus suggested that the axial core rotation is accelerated through the secondary circulation \( \Gamma_{CMB} \). The decadal conservation of the core-mantle angular momentum indicates that axial rotation of the mantle is accelerated by \( \Gamma_{CMB} \).

2.1.2. Computation of the topographic torque

We compute the resultant torque \( \Gamma_f \) by Hide’s method with the definitive topographic torque \( F \), since its error due to the approximation is only within the factor of \( \epsilon \) negligible compared with the uncertainty of \( \Gamma_f \) to be presented later (Section 4). Letting \( F \) denote \( \Gamma_f \) in Eq. (11), and \( \mathbf{u}_H \) denote \( \mathbf{u}_H(c) \) in Eq. (6), we now describe its computing procedure with spherical harmonics.

By expanding the dimensioned geostrophic pressure \( p_h^*(\theta, \phi) \) and topography \( h^*(\theta, \phi) \) in Schmidt quasi-normalized spherical harmonics \( Y_{m}^{\alpha}(\theta, \phi) \)

\[
p_h^* = \sum_{m=0}^{\infty} \sum_{a=1}^{\infty} \left( p_h^{m,c,s} Y_{m}^{c,s}(\theta, \phi) + p_h^{m,s} Y_{m}^{s}(\theta, \phi) \right),
\]

\[
h^* = \sum_{m=0}^{\infty} \sum_{a=1}^{\infty} \left( h^{m,c,s} Y_{m}^{c,s}(\theta, \phi) + h^{m,s} Y_{m}^{s}(\theta, \phi) \right),
\]

we can write Eq. (11) in terms of their coefficients \( p_h^{m,c,s} \) and \( h^{m,c,s} \) (see Greff-Lefftz and Legros (1995) for the explicit expression). It proves that the topographic torque \( F \) is given by a combination of coefficients \( h^{m,c,s} \) and \( p_h^{m,c,s} \)

\[
\Gamma_v = \mathbf{G}_{adp} \mathbf{h}_u \mathbf{p}_h
\]

where \( \mathbf{v} \) denotes the component of the torque \( \{x, y, z\} \), and \( \alpha \) and \( \beta \) denote indices of spherical harmonics \( n, m \) and \( \{c, s\} \). The elliptic shape of the CMB characterized by the core flattening \( f_c \) can be represented in terms of \( h^{m,c,s} \) to the first order approximation; it amounts to \( -2f_c \). The part of the equatorial torque related to the core flattening is denoted by \( \Gamma_{f,CMB} \) (Section 5.1).

The core surface flow \( \mathbf{u}_H \) is expressed by the expansion of its toroidal and consoidal functions in \( Y_{m}^{c,s}(\theta, \phi) \) up to truncation degree \( N_c \):

\[
\mathbf{u}_H = \sum_{m=0}^{N_c} \sum_{a=1}^{\infty} (a m)^{m,c,s} \mathbf{Y}_{m}^{c,s}(\theta, \phi) + \sum_{m=0}^{N_c} \sum_{a=1}^{\infty} (a m)^{m,s} \mathbf{Y}_{m}^{s}(\theta, \phi) + \sum_{m=0}^{N_c} \sum_{a=1}^{\infty} (a m)^{m,c,s} \mathbf{Y}_{m}^{c,s}(\theta, \phi) + \sum_{m=0}^{N_c} \sum_{a=1}^{\infty} (a m)^{m,s} \mathbf{Y}_{m}^{s}(\theta, \phi)
\]

(17)

2.2. The core surface flow

The core surface flow \( \mathbf{u}_H(c) \) is estimated from a geomagnetic field model and its temporal variation by applying the FF induction equation at the CMB (e.g. Bloxham and Jackson, 1991). Radial component of the equation

\[
\frac{\partial B_r}{\partial t} + \nabla \times (\mathbf{u}_H B_z) = 0,
\]

(23)

can be used to estimate \( \mathbf{u}_H \). Representing \( \mathbf{u}_H \) by Eq. (19) and \( B_z \) and \( B_r \) at the CMB conventionally by the Gauss coefficients, one can rewrite Eq. (23) in a matrix form

\[
\mathbf{b} = \mathbf{A} \mathbf{e} + \mathbf{c}
\]

(24)
where $b$ is a column vector that contains temporal variation of the Gauss coefficients $h^{\text{G}}_r^{(r-1)}$. The matrix $A$ contains the information of the MF. Errors of the SV coefficients would be obtained from Eq. (24) with the TG assumption imposed by the penalized maximum likelihood method (e.g. Holme, 1998; Pais et al., 2004). Nevertheless, we here restrict ourselves to Hide’s method using exact TG flows $u_H$ so as not to let the flow variance be even larger.

The flow still has unconstrained space even after the imposition of the TG assumption (e.g. Chulliat and Hulot, 2000). Any solution varying within such space will explain SV, out of which a most plausible one is usually selected by adding further constraints (see, for example, Pais and Hulot, 2000). Here, we use two kinds of additional constraint at one time in order to obtain a variety of core flow model solutions. One is the damping implemented by minimizing spatial roughness of solution flow

$$N_E = \int_{S} |u_H|^2 dS = E_E^{-1} \xi^T C_{bg}^{-1} \xi,$$

(26)

where $E_E$ is a non-dimensional scaling parameter and $C_{bg}^{-1} = \text{diag}((n+1)(2n+1)^{-1})$. The other is imposed likewise by minimizing kinetic energy of the flow

$$N_E = \int_{S} |u_H|^2 dS = E_E^{-1} \xi^T C_{bg}^{-1} \xi,$$

(27)

where $E_E$ is a non-dimensional scaling parameter and $C_{bg}^{-1} = \text{diag}((n+1)(2n+1)^{-1})$. The objective function to minimize is given by $M + N_E + N_E$, where $M$ is the weighted misfit norm $b - A_Q \xi$ and $N_E$, $N_E$ are obtained by

$$h = A_Q \xi$$

(25)

where $\lambda:\lambda = \sigma_{\xi v} E_E^{-1}$ and $\lambda_{\xi v} = \sigma_{\xi v} E_E^{-1}$ are damping parameters. And the covariance matrix for $\xi$ is obtained

Table 1
Combination of the topography and flow coefficients (up to degree 4)

| $r$ | topography | flow
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</tbody>
</table>

Circles indicate the combination corresponding to non-zero $E_0$ and $\sigma_{\xi v}$ as Eq. (22).
Attention should be paid to the argument by Celaya and less importance on minimizing checking especially for flows estimated with relatively otherwise the solution may incur aliasing even if dampable solution only if $l > 92$.

GSFC80 is necessary here both as a measure of acceptable level of fit by estimated flows (Section 4.1) and as a source of variance to be propagated numerically to that of the predicted topographic torque (Section 4.2). For the latter purpose, an absolutely scaled covariance matrix for the coefficients of the relevant SV model is needed. Weight matrix with unknown scaling employed in place of covariance matrix of a SV model (such as $W = \text{diag}(n + 1)$, Gire and Le Mouël, 1990) will not do in the present study.

We make a reasonable estimate of the SV covariance matrix $C_{SV}$ (e.g. $C_{SV} = \sigma_{SV}^2 \Sigma_C$, Section 2.2) of the model GSFC80 in the following way. By inverting observational data at 1980.0 for a temporary SV model $\hat{B}_{\text{tmp}}$, we obtain an estimate of the dimensionless matrix $C_{SV}$ which has non-zero off-diagonal elements reflecting inhomogeneous distribution of the variance of the SV model (Alexandrescu et al., 1994) due to the lack of magnetic stations around the oceans. The scaling parameter $\sigma_{SV}$ is then determined by referring to the disagreement of the existing SV models around 1980.0; as shall turn out in Section 3.1.1, this disagreement is significant. To show it first, the estimation of $C_{SV}$ is deterred until Section 3.1.2.

### 3. Models used in the analysis

#### 3.1. Geomagnetic secular variation and its variance

As a model of the SV to be used in our analysis we adopt the SV model of GSFC80 (Langel et al., 1982), the model at the epoch 1980.0 when geomagnetic data were augmented by the satellite MAGSAT. Though the SV model of GSFC80 has spherical harmonic coefficients up to degree 12, we have its truncation degree $N_S$ set at 8 in the following analyses. The harmonics below degree 8 are presumably attributed in the most part to the signal of the core origin, as opposed to the rest of the components which may be unjustified to be downward continued to the CMB due to the obscuring by the external signal (or the noise) that may compare the core signal. We consider that the ambiguity of the SV model due to the truncation is incorporated in its variance described below.

An estimate of the variance of the SV model of GSFC80 is necessary here both as a measure of acceptable level of fit by estimated flows (Section 4.1) and as a source of variance to be propagated numerically to that of the predicted topographic torque (Section 4.2). For the latter purpose, an absolutely scaled covariance matrix for the coefficients of the relevant SV model is needed. Weight matrix with unknown scaling employed in place of covariance matrix of a SV model (such as $W = \text{diag}(n + 1)$, Gire and Le Mouël, 1990) will not do in the present study.

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#### 3.1.1. Disagreements between the existing SV models around 1980.0

Comparing different SV models for similar epochs can provide a gross estimate of the variances of those models (e.g. Gire and Le Mouël, 1990). There are two SV models besides that of GSFC80 that are proposed as candidate for IGRF 1980, i.e. those of IGS80 (Barraclough et al., 1982) and USGS80 (Peddie and Fabiano, 1982). Defining $W(B)$ as rms magnitude of a field $B$ over the Earth's surface ($r = a$),

$$W(B) = \left( \frac{1}{4\pi a^2} \int \int \sum_{i=1}^{N} W_i(B)^2 \text{d}a \text{d}r \right)^{1/2} = \left( \sum_{i=1}^{N} W_{\sigma}(B) \right)^{1/2},$$

$$W_{\sigma}(B) = (n + 1) \sum_{A=0}^{A_{max}} \sum_{\alpha=0}^{\alpha_{max}} \left( \delta_{\alpha_{max}}^2 + \delta_{A_{max}}^2 \right),$$

we can demonstrate the significant difference between the SV models of GSFC80 $B_{\text{GSFC}}$ and USGS80 $B_{\text{USGS}}$ by calculating rms difference between them $W(B_{\text{GSFC}} - B_{\text{USGS}})$ (see Table 2) or that normalized by $W(B_{\text{GSFC}})$.
3.1.2. Estimating the covariance matrix

For a temporary SV model, the measurements for a SV model parameterized by temporal variation of Gauss coefficients are inverted for a SV model with coefficients only of internal origin up to degree N = 8. We thus obtain a temporary model field Btmp which explains 303 data with average misfit 0.254 nT year\(^{-1}\). This average misfit is smaller by an order of magnitude than the expected observation uncertainty of the SV (~4 nT year\(^{-1}\)). Barker and Barracough (1985), so the model Btmp may be overfitted to the data. Table 3 shows the lower degree coefficients of Btmp and BGSFC. The disagreement of these models is comparable to W(BGSFC - BUSGS) (Table 2).

3.1.2. Estimating the covariance matrix ΣSV of the SV model of GSFC80

Now, let’s make a reasonable estimate of the covariance matrix for Btmp at 1980.0, which we shall apply to obtain ΣSV, by starting with the inversion of observations for a temporary SV model Btmp (parameterized by temporal variation of Gauss coefficients btmp). Out of the magnetic stations which provide annual record of three components of geomagnetic data open at the website of National Geophysical Data Center (http://www.ngdc.noaa.gov/), we select a dataset by picking up as many stations (operating for period 1959.5–1991.5) as possible, while the site that has missing data for more than 5 years is rejected. One hundred and one sites are thus extracted (Fig. 1); 79 and 22 are located in the northern and southern hemispheres, respectively. After calculating annual differences of each component of each site, we obtain SV time series data by calculating their 5-year running average. Following the procedure of stochastic inversion for a SV model by Gubbins (1983), the SV data from all 101 sites at 1980.0 (303 data in total, their covariance matrix being assumed to be proportional to the identity matrix) are inverted for a SV model with coefficients btmp of only internal origin up to degree N = 8. Thus we obtain a temporary model field Btmp which explains 303 data with average misfit 0.254 nT year\(^{-1}\). This average misfit is smaller by an order of magnitude than the expected observation uncertainty of the SV (~4 nT year\(^{-1}\)). Barker and Barracough (1985), so the model Btmp may be overfitted to the data. Table 3 shows the lower degree coefficients of Btmp and BGSFC. The disagreement of these models is comparable to W(BGSFC - BUSGS) (Table 2). The disagreement of these models is comparable to W(BGSFC - BUSGS) (Table 2).

Now, the covariance matrix of btmp—to be applied to Ctmp—is obtained by propagating the covariance (characterized by the unscaled identity matrix) of the magnetic data (leading to Cb), and by calculating unbiased estimator for σtmp\(^2\) (denoted temporarily by σtmp\(^2\)) with the help of a posteriori misfit (σtmp\(^2\) = 25.2 nT\(^2\) year\(^{-1}\)). Spatial distribution of variance of the SV model thus estimated can be shown, for example, by plotting the standard deviation σ(btmp(r)) of radial component of the predicted field Btmp(r) at an arbitrary position r on the Earth’s surface (Fig. 1). It is computed by

σ\(^2\) (btmp(r)) = a(tmp(r)|Ctmp) / a\(^2\) (r|Ctmp).

where a\(^2\) (r|Ctmp) is a row vector relating model parameters btmp to the predicted field Btmp(r), i.e. btmp(r) = a\(^2\) (r) / a\(^2\) (r|Ctmp) (e.g. Alexandrescu et al., 1994).

As a final step to determine the matrix Ctmp for BGSFC, we select an appropriate scaling for it by referring to the

---

Table 2

<table>
<thead>
<tr>
<th>Component</th>
<th>Rms magnitude</th>
<th>Rms difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>ith site</td>
<td></td>
<td></td>
</tr>
<tr>
<td>jth site</td>
<td></td>
<td></td>
</tr>
<tr>
<td>kth site</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

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Fig. 1. Distribution of the 101 magnetic stations (solid triangles) whose data are used to construct the model Btmp (see text). Solid lines are the standard deviation of radial component of the model Btmp at the Earth’s surface σ(btmp(r)_R)). Contour interval is 1 nT year\(^{-1}\).
Table 3: Spherical harmonic coefficients of lower degrees of the models \( B_{\text{GSFC}} \) by Langel et al. (1982) and \( B_{\text{USGS}} \) estimated in this study (second and third columns).

<table>
<thead>
<tr>
<th>Degree</th>
<th>( B_{\text{GSFC}} )</th>
<th>( B_{\text{USGS}} )</th>
<th>( \Delta B )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>20.51</td>
<td>23.47</td>
<td>0.87</td>
</tr>
<tr>
<td>1</td>
<td>9.08</td>
<td>10.02</td>
<td>0.72</td>
</tr>
<tr>
<td>2</td>
<td>19.53</td>
<td>16.90</td>
<td>0.64</td>
</tr>
<tr>
<td>3</td>
<td>4.78</td>
<td>4.23</td>
<td>0.69</td>
</tr>
<tr>
<td>4</td>
<td>18.79</td>
<td>12.69</td>
<td>0.72</td>
</tr>
<tr>
<td>5</td>
<td>9.66</td>
<td>5.12</td>
<td>0.69</td>
</tr>
<tr>
<td>6</td>
<td>26.82</td>
<td>24.09</td>
<td>0.75</td>
</tr>
<tr>
<td>7</td>
<td>4.51</td>
<td>2.65</td>
<td>0.55</td>
</tr>
<tr>
<td>8</td>
<td>3.55</td>
<td>5.65</td>
<td>0.60</td>
</tr>
<tr>
<td>9</td>
<td>5.67</td>
<td>4.77</td>
<td>0.66</td>
</tr>
<tr>
<td>0 1</td>
<td>2.19</td>
<td>0.82</td>
<td>0.58</td>
</tr>
<tr>
<td>1 2</td>
<td>1.64</td>
<td>2.11</td>
<td>0.73</td>
</tr>
<tr>
<td>2 3</td>
<td>3.87</td>
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</tr>
<tr>
<td>3 4</td>
<td>−6.62</td>
<td>−7.58</td>
<td>0.58</td>
</tr>
<tr>
<td>4 5</td>
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<td>−0.38</td>
<td>0.54</td>
</tr>
<tr>
<td>5 6</td>
<td>−2.4</td>
<td>−1.57</td>
<td>0.51</td>
</tr>
<tr>
<td>6 7</td>
<td>3.66</td>
<td>2.95</td>
<td>0.57</td>
</tr>
<tr>
<td>7 8</td>
<td>−10.09</td>
<td>−7.89</td>
<td>0.46</td>
</tr>
<tr>
<td>8 9</td>
<td>1.08</td>
<td>0.92</td>
<td>0.70</td>
</tr>
<tr>
<td>9 0 1</td>
<td>−3.82</td>
<td>−2.51</td>
<td>0.58</td>
</tr>
<tr>
<td>1 2 3</td>
<td>6.77</td>
<td>3.13</td>
<td>0.55</td>
</tr>
<tr>
<td>2 3 4</td>
<td>−5.54</td>
<td>−6.06</td>
<td>0.60</td>
</tr>
<tr>
<td>3 4 5</td>
<td>−2.73</td>
<td>−1.17</td>
<td>0.54</td>
</tr>
</tbody>
</table>

Fourth column is estimated standard deviation of \( \Delta B \) calculated with unbiased estimator \( \sigma_{\text{bias}} \), i.e., \( \sigma_{\text{bias}} = \sqrt{\text{bias}(\sigma_{\text{bias}})^2} \). All values are in nT year\(^{-1}\).

rms differences of the existing models (Section 3.1.1).

Standard deviation \( \sigma(W(\dot{B}_{\text{GSFC}})) \) of rms magnitude of the temporary field \( \dot{B}_{\text{GSFC}} \) can be computed in the same way as Eq. (32), but with a column vector \( W(\dot{B}_{\text{GSFC}})/\dot{B}_{\text{GSFC}} \) substituted for \( \mathbf{a}/(\mathbf{r}) \); it turns out to be 1.03 nT year\(^{-1}\). This is smaller than \( W(\dot{B}_{\text{GSFC}} - \dot{B}_{\text{USGS}}) \) by more than one order of magnitude, indicating that our estimate of \( \sigma_{\text{bias}} \) can be quite optimistic. Using this estimate in the inversion for the flow (Section 2.2) would force the solution to be overfitted to the SV model \( \dot{B}_{\text{GSFC}} \). For the definitive estimate of \( \sigma_{\text{bias}} \), we modify \( \sigma_{\text{bias}} \) by multiplying it by \( 10^2 \) (i.e., \( \sigma_{\text{bias}} = 10^2 \sigma_{\text{bias}} = 10^2 \times 1.03 \text{nT}_\text{year}^{-1} \)), thereby letting \( \sigma(W(\dot{B}_{\text{GSFC}})) = 10.3 \text{nT}_\text{year}^{-1} \), comparable to the order of \( \sigma(W(\dot{B}_{\text{GSFC}} - \dot{B}_{\text{USGS}})) \). On the basis of \( \dot{C}_{\text{SV}} \) with this scaling, a set of statistically deviated coefficients from those of the model \( \dot{B}_{\text{GSFC}} \) will lead to a SV model with gross disagreement of approximately \( \pm \sigma_{\text{bias}} \) with \( \dot{K}_{\text{USGS}} \).

There is a pragmatic estimate of the SV variance in terms of degrees \( n \) and epochs (rotationally invariant errors in nT year\(^{-1}\), Jackson, 1997), to which Pais and Hulot (2000) referred to scale the weight matrix \( \hat{W} = \text{diag}(n + 1) \) employed in place of a SV covariance matrix in their flow inversion. Their corresponding standard deviations are smaller by an order of magnitude than the rotationally invariant errors calculated likewise from our \( \dot{C}_{\text{SV}} \) (Fig. 2). But we note the implication by Celaya and Wahr (1996) which reads as that, if the elevated flow powers in higher degrees appearing when the flow is estimated without dampings are to be ascribed to the SV noise, the SV standard deviation should be larger by at most an order of magnitude than the level of rotationally invariant errors of Jackson (1997). We would rather consider here the ambiguity of the SV can actually be greater than what is indicated by the estimated variance of individual SV models, and view our \( \dot{C}_{\text{SV}} \) as the upper bound of the SV variance.
3.2. The main field

Owing to the satellite MAGSAT, the model DGRF-1980 seems to achieve higher reliability than other DGRF models. The truncation degree $N_{MF}$ of its MF model $B_{DGRF}$ is 10, and we make use of all its coefficients up to degree 10 to calculate the matrix $A$ in Eq. (24). This calculation implicitly regards that at the CMB there is no effective contribution of still higher degree components, which, along with a reasonable behavior of the SV spectrum at the CMB, might be sufficient for the flow to be of large-scale (this could be true unless any significance is attached to the power of the MF higher than degree 13 at the CMB, see Chulliat and Hulot, 2000).

The MF model $B_{DGRF}$ can possibly be associated with variance due to the truncation, i.e. aliasing by the higher degree field (such as due to the crustal anomaly). Moreover, it can be contaminated by measurement errors. These variances would attach additional ambiguity to the estimate of the flow $\zeta$, rendering the inversion too complicated (probability density function to maximize becomes intractable, Jackson, 1995). For the present, nevertheless, the error of $B_{DGRF}$ is neglected, being regarded as insignificant compared with that of the SV model, which reduces the estimation to be simple linear inversion problem as in Section 2.2.

3.3. The CMB topography

Many previous seismological studies have attempted to image the map of the CMB topography (e.g. Boschi and Dziewonski, 2000; Sze, 2003). The estimated models are poorly correlated with one another, depending on the employed phases or model parameters to deduce them, except for their largest-scale patterns that may have relatively prominent power at degree 2. In fact, even the typical amplitude of the topography is not clearly resolved; the models by Boschi and Dziewonski (2000) have a peak-to-peak amplitude of over 10 km, while the estimate by Sze (2003) yields that of only 3 km. The small-scale topography can be subject to more ambiguity.

For the CMB topography to be used in the present analysis we adopt the model 8 by Boschi and Dziewonski (2000) which is given by grid data. The grid is arranged such that its spacing are $5^\circ \times 5^\circ$ in the neighborhood of the geographic equator and nearly equal areas are assigned to its squares covering the whole spherical surface. The spherical harmonic coefficients $h^m_c(s)$ (Section 2.1) are obtained by computing the spherical transform of their model. Its image and power spectrum are shown in Figs. 3 and 4, respectively.

It is obvious from Eq. (11) that the torque calculated by Hide’s method varies in proportion to the scale of topography amplitude $\epsilon$. But, the dependence of the torque on the topography emerges rather from minor modification of its morphology resulting in a drastic variation of the subsequent torque (Jault and Le Mouël, 1990); indeed, topography models are fraught with such large variance (perhaps the uncertainty is by far greater than that of core surface flow) that it is apparently unable to discern the minor modification by seismic observations.

We consider the adopted CMB topography as a reference model which can basically be perturbed in a systematic study, while its power spectrum (Fig. 4) is presumed at least robust. To make the present argument clear, nevertheless, we here set the topography model fixed with no variance, and focus primarily on the uncertainty of the topographic torque due only to that of the flow.
4. Uncertainty of the topographic torque

We now investigate the variability of the topographic torque by using core surface flow models and their variances estimated from the geomagnetic models, $B_{\text{GSFC}}$ and $B_{\text{BGRS}}$, and the covariance matrix $C_{\text{GSFC}}$. According to Eq. (23) and the selection rule of Gaunt and Elsasser integrals, the MF and SV (truncated at degrees $N_{\text{MF}}$ and $N_{\text{SV}}$, respectively) are related to the flow of degree up to $N_{\text{MF}} + N_{\text{SV}} (+18$, if $N_{\text{MF}} = 10$ and $N_{\text{SV}} = 8$). We still select 14 as the truncation degree of the flow $N_u$ which may correspond to the smallest scale that can be resolved (Bloxham and Jackson, 1991). Yet, components higher than degree 8 are thought to be scarcely constrained by geomagnetic data (Hulot et al., 1992) and hence would be determined mainly by the dampings in the actual inversion. In Eq. (22), it can be seen that the flow truncated at degree $N_u$ interacts with the CMB topology up to degree $N_{\text{TP}} = N_u + 1$ to generate the topographic torque ($N_{\text{TP}} = 15$ if $N_u = 14$).

The cause that gives rise to the uncertainty of estimated flows is twofold. One is the non-uniqueness problem of the flow inversion and the other is the variance of the SV model (including the possible aliasing in the SV model due to the truncation). Since they are based on independent problems (i.e. theoretical and observational), their influences on the subsequential topographic torque should be treated separately. We address the variation of the torque due to the non-uniqueness in Section 4.1, and then the variance of the torque originating in that of the SV in Section 4.2. In both sections, errors of the MF and the CMB topology are neglected, as noted earlier in Sections 3.2 and 3.3.

4.1. Variation of the topographic torque due to the non-uniqueness of core surface flow inversion

The space of core surface flow $u_H$ contains a subspace which is not constrained by the geomagnetic data nor by the assumption of TG. Analytically, the elements of this subspace are the flows having the stream function $\psi(\theta, \phi) \equiv B_\ell / \cos \theta$ within the ambiguous patch (Backus and Le Mouël, 1986; Chulliat and Hulot, 2000). In this paper, flows within such subspace, $u_\psi$, producing no SV are referred to as the invisible flow linear algebraically, this subspace would correspond to the null-space or kernel of the matrix $A_{\phi}$ (which depends only on the MF model), i.e. $A_{\phi} u_{\psi} = 0$ where $u_{\psi}$ denotes the vector of the TG coefficients of $u_{\psi}$, the invisible flow on spatial scales of interest (with its degrees not higher than $N_u$). The dimensions of the column and row of the matrix $A_{\phi}$ are $N_{SV}(N_{SV} + 2) = 80$ and $N_u^2 = 196$, respectively, when $N_{SV} = 8$ and $N_u = 14$.

This means that the null-space does not always exist simply because $\text{Ker}(A_{\phi}) \supset N_{SV}(N_{SV} + 2) - N_u^2 > 0$ as well as because the matrix is ill-conditioned. Of course, $u_{\psi}$ does not necessarily coincide with the invisible flow $u_{\psi}$ of the analytical sense, since $u_{\psi}$ is truncated and does not give a perfect description of $u_{\psi}$. (Note here that the term invisible flow can alternatively indicate a part of the true core surface flow that cannot be retrieved by the inversion (Eymin and Hulot, 2005). In this case, the invisible flow would lie in the kernel of the resolution matrix $R(\omega H_{\text{A}})$.)

The topographic torque by Hide’s method may vary in accordance with the variation of $u_{\psi}$. For the present purpose, therefore, it is needed to seek various reasonable $u_{\psi}$ out of its mathematically infinite families. To that aim, the flow inversion (Eq. (28)) is performed with a broad range of damping parameters $\lambda_E$ and $\lambda_R$ which control the abundance and distribution of $u_{\psi}$. In practice, $\lambda_E$ and $\lambda_R$ are varied within the range from $10^{-3}$ to $10^2$ and from $10^{-1}$ to $10^2$, respectively.

4.1.1. Misfit to the SV model by the estimated flows

The estimated flows vary as the values of the parameters $\lambda_E$ and $\lambda_R$ are changed. Illustrated in Fig. 5 is square shaded area in Fig. 5, since the effect of changing the values of $N_{\text{SV}}$ on the flow solution is on the flow solution is not independent of each other.

The condition $\chi < \chi_C$ (Eq. (31)) may be useful to qualify the obtained flow models as explaining the SV model $B_{\text{GSFC}}$. This assessment is made by using the predicted SV only up to the truncation degree of $B_{\text{GSFC}}$ (i.e. $N_{\text{SV}} = 8$), though flow and MF models truncated at degree $N_u$ and $N_{\text{MF}}$, respectively, generate SV up to degree $N_u + N_{\text{MF}} (+24$, if $N_u = 14$ and $N_{\text{MF}} = 10$). Indeed, in Eq. (33) the predicted SV above degree 8 is regarded as of little significance, because such part of SV makes up no more than 0.6% of the total energy of predicted SV over the Earth’s surface.

For the convenience of understanding the variability of obtained flows and accompanying topographic torques, we here take four examples of flow model A–D such that all their misfit $\chi$ are equal to 0.30 (see Fig. 5).
Fig. 5. Misfit $\chi$ of the estimated flows in $\sqrt{\text{ERNR}}$ (rms roughness) − $\sqrt{\text{EENE}}$ (rms velocity) plane. $\sqrt{\text{ERNR}}$ and $\sqrt{\text{EENE}}$ are determined by the damping parameters $\lambda_R$ and $\lambda_E$, respectively. Those of the flow models A–D shown in Fig. 6 are indicated by +. Contour interval is 0.05.

Their properties and images are presented in Table 4 and Fig. 6, respectively. All their rms velocities ($\sqrt{\text{EENE}}$) are not unrealistic when compared to the rate of geomagnetic westward drift ($\sim$15 km year$^{-1}$).

4.1.2. Variation of the estimated TG flows

We try to show in this subsection how the variation of obtained flows as seen in Figs. 5 or 6 is caused while their compatibility is kept with both the SV model and the assumption of TG. We then check if the obtained flows are reasonable enough to ensure the convergence of their power spectra below the truncation degree $N_u = 14$ (as claimed in Section 2.2).

The variation of the obtained flows as shown in Fig. 5 is expected to result from that of $u_{\text{null}}$. Here, we note that, in the practical flow inversion as the present study, $u_{\text{null}}$ may be interpreted such that it is recognized as the part of flow that generates no more significant SV than the error of the SV model, as long as $\chi < \chi_C$. In this view, the definition of $\zeta_{\text{null}}$, namely the flow coefficients of $u_{\text{null}}$ satisfying $A_{tg} \zeta_{\text{null}} = 0$, may be

Table 4
Properties of the flow models A–D

<table>
<thead>
<tr>
<th>$\lambda_R$ [nT$^2$ year$^{-2}$]</th>
<th>$\sqrt{\text{ERNR}}$ [km year$^{-1}$]</th>
<th>$\sqrt{\text{EENE}}$ [km year$^{-1}$]</th>
<th>$\Gamma_x$ [$10^{19}$ Nm]</th>
<th>$\Gamma_y$ [$10^{19}$ Nm]</th>
<th>$\Gamma_z$ [$10^{19}$ Nm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2.0 × 10$^{-4}$</td>
<td>432.9</td>
<td>17.72</td>
<td>−5.74</td>
<td>−0.70</td>
</tr>
<tr>
<td>B</td>
<td>1.2 × 10$^{-4}$</td>
<td>481.0</td>
<td>13.76</td>
<td>−4.53</td>
<td>−0.30</td>
</tr>
<tr>
<td>C</td>
<td>4.6 × 10$^{-2}$</td>
<td>621.9</td>
<td>11.13</td>
<td>−3.57</td>
<td>−0.30</td>
</tr>
<tr>
<td>D</td>
<td>0.0</td>
<td>892.6</td>
<td>10.31</td>
<td>−3.28</td>
<td>−0.70</td>
</tr>
</tbody>
</table>

Fig. 6. Image of the flow models A–D (see Table 4). Continents are those projected onto the CMB.
Fig. 7. Power spectra $E(n)$ of the flow models A–D. The standard deviations of each $E(n)$ given by error bars are obtained by the Gaussian error propagation law using the solution $\hat{\zeta}$ (Eq. (28)) and its covariance matrix $C_{\hat{\zeta}}$ (Eq. (29), i.e. based on the variance of the SV model scaled absolutely in Section 3.1.2. See also Section 4.2.1).

We refer to those flows that satisfy Eq. (34) as the extended invisible flow.

We confirm that there is an evident trade-off relation between larger and smaller scales of the flow power along the contour of the misfit $\chi$ in Fig. 5. This feature is attributed naturally to the intention of the dampings (Section 2.2); minimizing $N_R$ (corresponding to the model
A) damps higher degrees more effectively than \( \mathcal{N}_E \) (corresponding to the model D). As for the models A–C, the spectra \( E(n) \) seem to decrease as the degree \( n \) increases and to achieve the convergence. On the other hand, the spectrum of the model D, obtained with the damping by \( \mathcal{N}_E \) alone, exhibits no sign of convergence below degree \( \mathcal{N}_u \). It is indicated that the model D (and those estimated with little importance on minimizing \( \mathcal{N}_R \)) can be disrupted by serious aliasing due to inappropriate truncation, if the spectrum of the true flow converges above degree \( \mathcal{N}_u \). Bearing this in mind, nonetheless, we investigate the torque with respect to all the obtained flows.

4.1.3. Variation of the calculated topographic torques

Presented in Fig. 9 is the axial topographic torque \( \Gamma_z \) calculated using the flows of which misfits \( \chi \) are shown in Fig. 5. It has typical magnitude of order \( 10^{18} \) to \( 10^{19} \) Nm with both signs. This result simply implies the possibility for the net torque to be reduced to the suitable value for the decadal LOD variations, if the torque is realized around the transition area of its sign in Fig. 9. The misfit of such flows in the present analysis are small enough to explain the SV model \( \dot{\mathbf{B}}_{\text{GSFC}} \) (Fig. 5), whereas their rms roughness \( \sqrt{\langle E_{r} \rangle_{\text{rms}}} \) are required to be larger than those flows in Fig. 6. To check the dynamical consistency including the core, further investigation will be required (Section 5.3).

The equatorial components of the torque, \( \Gamma_x \) and \( \Gamma_y \), obtained in the same way are shown in Fig. 10. Both components have the magnitude and variability of order \( 10^{19} \) Nm, larger by an order of magnitude than those of \( \Gamma_z \). Unlike the case with \( \Gamma_z \), they tend to vary along the contour of the misfit \( \chi \) in Fig. 5; their magnitude increases with the growth of flow velocity along that contour and consequently of the amplitude of large-scale geostrophic pressure. To this point, we shall figure out that the flattening of the CMB is the most relevant (Section 5.1).

4.2. Variance of the topographic torque associated with that of the geomagnetic SV

Since we do not in practice have perfect knowledge of the geomagnetic SV, the maximum likelihood solutions of TG flow \( \hat{\zeta} \), and their concomitant topographic torques, might bear some errors that originate in that of geomagnetic SV model. We now turn to examine the ambiguity of the torque by evaluating their possible variances. The source of variance is supposed to lie only in that of the SV model \( \dot{\mathbf{B}}_{\text{GSFC}} \) up to degree \( \mathcal{N}_V = 8 \), as given by the absolutely scaled covariance matrix \( C_{SV} \) obtained in Section 3.1.2.

4.2.1. Variance of the estimated TG flows

The standard deviation of core surface flow (for both components, \( u_\theta \) and \( u_\phi \)) is calculated from the covariance matrix of its coefficients \( C_{\hat{\zeta}} \) obtained by Eq. (29) where \( 2.52 \times 10^{-3} \text{ m}^2 \text{ year}^{-1} \) is used for \( \sigma_{SV}^2 \) (Section 3.1.2).
Fig. 10. Equatorial components of the predicted topographic torque on the mantle, $\Gamma_x$ and $\Gamma_y$, associated with the same flows as in Fig. 5. Contour interval is $5 \times 10^{19}$ Nm. Solid and dashed curves correspond to positive and negative values of the torque, respectively.

Fig. 11 presents the flow model A and its standard deviation. It turns out that in some regions the error of the flow is not smaller than the local flow there. The flow error appears to be relatively larger beneath the region where the observational uncertainty of SV is significant such as in the southern Pacific (Fig. 1); this spatial variation of the flow error would be more prominent if the solution $\xi$ is obtained with less influence by the dampings, as they work on the variance of solution in the same senses as on the solution itself. Also noticeable is that the flow error tends to be larger in the direction along the contour of $\psi(\theta, \phi)$ (Section 4.1), especially where $|B_r|$ is small. This spatial distribution of the error is in line with that of the invisible flow (Section 4.1.2), reflecting the nature of the inversion that TG flow along the contour of $\psi(\theta, \phi)$ within the ambiguous patch is unconstrained by geomagnetic data, and hence determined only by the dampings.

4.2.2. Variance of the calculated topographic torques

Fig. 12 shows the possible dispersion of $\Gamma_x$, $\Gamma_y$ and $\Gamma_z$ from the torque associated with the solutions of maximum likelihood $\hat{\xi}$ which is obtained by Eq. (28) with $\lambda_E$ kept 0 and $\lambda_R$ changed; these solutions correspond to those along the edge of shaded area in the side of larger rms flow velocity ($\sqrt{E_{N\xi}}$) in Fig. 5, and are ensured to converge below the truncation degree $N_a = 14$. With this series of solutions, the condition $\chi < \chi_C$ (Section 4.1.1) is satisfied as long as the $\sqrt{E_{N\xi}}$ is greater than 16.3 km year$^{-1}$. The dispersed torques in Fig. 12 are calculated using the flows whose coefficients are statistically deviated from $\hat{\xi}$; here, such coefficients are selected randomly and repeatedly so as to constitute the Gaussian distribution with mean values $\hat{\xi}$ and with the variances given by the diagonal components of the matrix $C_{\hat{\xi}}$.

The variance of the topographic torque due to that of the SV model becomes greater as the flow velocity $\sqrt{E_{N\xi}}$ is increased, or the damping parameter $\lambda_R$ is decreased. This is attributed simply to the magnitude of variance of the flow solutions $\xi$; those with larger $\sqrt{E_{N\xi}}$ explain the SV with smaller misfit $\chi$, and that is to say, have more margin for the variance till it reaches the level of the critical misfit $\chi_C$. The scaling of the matrix $C_{\xi}$ has been regulated so that the misfit $\chi$ of the deviated flows created as described above in this subsection statistically becomes at the same level as $\chi_C$ (see Section 3.1.2).

It is clear in Fig. 12 that the variance is larger for $\Gamma_x$ and $\Gamma_y$ (of order $10^{20}$ Nm) than for $\Gamma_z$ (of order $10^{19}$ Nm). Compared with the variation of each component of the torque due to the non-uniqueness of the flow inversion in Section 4.1, the variance of the torques associated with that of the flow (or originally of the geomagnetic SV) is larger by an order of magnitude; the standard deviation is of the order $10^{19}$ Nm for $\Gamma_z$ and
Fig. 12. Variance of predicted topographic torque associated with variance of estimated core surface flow ($\Gamma_z$, top), $\Gamma_x$ (middle) and $\Gamma_y$ (bottom). Horizontal axis is rms velocity $\sqrt{E_{NN}}$. Solid line is the torque due to the flow of highest likelihood with $\lambda_E$ kept 0 and $\lambda_R$ varied. Dashed line is its standard deviation. Dots are the torques due to perturbed flows.

The net topographic torque calculated by Hide’s method in the previous section definitely indicate that they are subject to significant ambiguity and variance. To investigate how this uncertainty is caused, we make a scale-by-scale examination of the contribution of CMB topography to the net torque $\Gamma_{\nu}$ ($\nu = \{x, y, z\}$). The topography of each spherical harmonic mode can interact with specific modes of TG flow (see Table 1), so we can obtain the torques due to different scales of the topography by taking the sum of all elemental torques related to the topography of specified degrees. Let’s define $\Gamma_{\nu}^{(n_k)}$ as the torque associated with the topography of degree $n_k$ (i.e. $\Gamma_{\nu} = \Gamma_{\nu}^{(f_c)} + \sum_{n_k} \Gamma_{\nu}^{(n_k)}$, where $\Gamma_{\nu}^{(f_c)}$ is the equatorial torque related to the flattening of the CMB).

Fig. 13 shows $\Gamma_z^{(n_k)}$ and their estimated errors (from $\hat{\Gamma}_z$) calculated with respect to four flow models, A–D. In all cases, the net torque $\Gamma_z$ seems attributed mainly to $\Gamma_z^{(n_k)}$ with $2 \leq n_k \leq 6$. In particular, owing to the comparatively great power of the topography at degrees 2 and 3 (Fig. 4), the largest contribution to $\Gamma_z$ arises from $\Gamma_z^{(2)}$ and $\Gamma_z^{(3)}$, whereas the effect of the topography (especially $\Gamma_y^{(3)}$) cannot be disregarded. In fact, this is the reason why the variations of $\Gamma_x$ and $\Gamma_y$ along the contour of $\chi$ (Fig. 5) stand out in Fig. 10; the magnitude of $\Gamma_y^{(3)}$ grows as the flow power at lower degrees increases from the model D to A (Fig. 7). This feature can be explained qualitatively by comparing the global maps of the flow models (see Fig. 6); note the variability of the amplitude of large-scale geostrophic pressure anomaly (acting on the elliptic...
Fig. 13. The axial torque associated with specific degrees $n_h$ of the CMB topography by Boschi and Dziewonski (2000). The flow models A–D are employed (from top to bottom).

CMB) at mid-latitudes, such as a positive peak associated with the characteristic retrograde vortical flow beneath southern Africa.

While it is apparent that the large-scale interaction between the flow and topography is relevant to both the magnitude and uncertainty of $\Gamma_n$, potential contribution of the smaller-scale interaction has yet to be confirmed. For the flow models A and B, the calculated $\Gamma(n_h)$ and their errors with regard to the topography of higher degrees ($n_h > 10$) are much smaller than those with the topography of lower degrees, implying that there is little contribution of the smaller-scale topography. On the other hand, $\Gamma(n_h)$ with smaller-scale topography cannot be neglected for the flow model D (which has considerable power at higher degrees).

One may be able to ascertain the above argument by a simple statistical approach. On the basis of the formula of the topographic torque in terms of the coefficients of flow $c^{(c,s)}_n$ and $s^{(c,s)}_n$ and topography $h^{(c,s)}_n$ (Section 2), a rough estimate of the possible magnitude of the torque $\Gamma^{(c,s)}_n$ may be made; if effective cancellation is not to occur, it is at most proportional to $n_h^2 h_{n_h}^{-\frac{1}{2}}$, where $h_{n_h}$ and $h_{n_h}$ are the typical magnitude of flow and topography coefficients at degree $n_h$. The typical flow coefficients $h_{n_h}$ tend to behave as $\propto n_h^{-2.5}$ (model A) and $\propto n_h^{-2.5}$ (model D), when they are estimated with the damping using the norms $N_R$ and $N_E$, respectively. The behavior of the typical coefficients $h_{n_h}$ of the presently adopted topography is reasonably found $\propto n_h^{-1}$, whereas this fails to fit the anomalously small magnitude at degree 1 (see Fig. 4). If the flow spectrum exhibits such convergence as the model A, then $\Gamma(n_h) \propto n_h^{-1.5}$ and discounting of the topographic torque due to smaller-scale structures may be permitted. In contrast, if the true flow resembles the model D with still smaller-scale flows continuing beyond the degree $N_w$, we have $\Gamma(n_h) \propto n_h^{-2.5}$ and may have to expect critical contribution of smaller-scale topography to the net torque, as discussed by Jault and Le Mouël (1990). With large-scale topography models which has smaller amplitude, e.g. those with the peak-to-peak amplitude no more than 3 km (Sze, 2003), one may possibly find that $h_{n_h} \propto n_h^{l > 0}$; in this case, nevertheless, the smaller-scale interaction is unimportant as long as it is associated with the flow model A, but other flow models are not free from the possibility that non-negligible
contribution to the torque arises from smaller-scale structures.

5.2. The topographic coupling and the decadal polar motion

In the past studies, the calculations of equatorial topographic torque by Hide’s method provided no convincing result for explaining the Markowitz wobble (Section 1); no meaningful agreement has been found between the observed and predicted mean pole in terms of both its amplitude and phase. Now that the calculated equatorial torque is revealed to be widely variable, we reexamine it by calculating the torque considering its uncertainty studied in Section 4.

Displacement of the mean pole on the tangent plane at the North pole with the axes along the 0° and 90°E
meridians, \((m_1, m_2)\), is predicted by
\[
m = \frac{i}{\sigma_{\text{ew}}} \sum_i \Gamma_i
\]
where \(i = \sqrt{-1}\), \(m = m_1 + im_2\), \(\sigma_{\text{ew}}\) is the Chandler frequency and \(\sum_i \Gamma_i\) is the equatorial moment of inertia of the mantle (Hulot et al., 1996). \(\Gamma_i = \Gamma_{i1} + \Gamma_{i0}\) is the effective equatorial torque for which the deformation of elastic mantle due to the pressure from the liquid core is taken into consideration; it is estimated that the deformation of the mantle reduces the effect of \(\Gamma_{i1}\) by an order of magnitude (Greff-Lefftz and Legros, 1995; Hulot et al., 1996). So we obtain \(\Gamma_{i1}\) from \(\Gamma_{i1}\) by replacing \(\Gamma_{i1}\) with \(0.1\Gamma_{i1}\) (i.e. \(\Gamma_{i1} = 0.1\Gamma_{i1}\) \(\sim \sigma_{\text{ew}}^2\) \(\Gamma_{i1}\)).

Presented in Fig. 15 are amplitude (\(m_1^2 + m_2^2\))^1/2 and phase \(\tan^{-1}(m_2/m_1)\) of the mean pole and their possible dispersion associated with the flows obtained in Section 4 (flows estimated with \(\lambda_E = 0\)). The amplitude is typically less than \(\sim 10\) milliarcseconds, as long as the rms flow velocity \(\sqrt{ER_{\text{ne}} \lambda_E}\) is limited below \(\sim 20\) km year\(^{-1}\). With larger \(\sqrt{ER_{\text{ne}} \lambda_E}\), some of perturbed flows attain greater amplitude amounting to that of the observed oscillation of the Markowitz wobble in recent years (\(\sim 20\) milliarcseconds around 1980). The phase is typically limited within the range between \(-90^\circ\) and \(30^\circ\); this selectivity of the phase is not in good agreement with the observed mean pole.

As noted in Section 5.1, the amplitude and phase of the net equatorial torque \(\Gamma_{i1}\) and of the subsequent mean pole \(m\) (Eq. (36)), are affected predominantly by \(\Gamma_{(2)}\) and \(\Gamma_{(3)}\), the torques related to the topography of degrees 2 and 3. It is due to the insufficient magnitude (and variance) of these torques that the amplitude of possible \(m\) is inadequate for the observation when \(\sqrt{ER_{\text{ne}} N_{\text{e}}} < 20\) km year\(^{-1}\). To predict the preferred mean pole for the observation, therefore, it is necessary that flows of lower degrees hold larger energy (substantial part of which would be those unconstrained by geomagnetic data) than those of models estimated with the prior beliefs to adequately reduce the flow roughness or energy. Moreover, to predict the observed oscillation, distribution and energy of such flows should vary drastically with the periodicity of approximately 30 years, which is unrealistic. It may not be possible to create a flow model consistent with both the observed SV and wandering of the mean pole, but our results rather support the findings by Greff-Lefftz and Legros (1995), that is, it is unlikely for the Markowitz wobble to be excited exclusively by the topographic coupling.

5.3. The topographic coupling and the constraint on flow models

The topographic torque may be used as an additional constraint in estimating a model of the core surface flow, by choosing its particular configuration explaining both the SV and LOD observations. According to the results presented in Section 4, however, unresolved space of the flow by geomagnetic data is too large to be determined thoroughly by this constraint, as is apparent from the irrelevance of the axial topographic torque \(\Gamma_1\) to the zonal toroidal flow coefficients \(\lambda_1^m\) (Table 1). Referring to the theory of core dynamics may further narrow the range of the possible flow; even if a torque \(\Gamma_i\) associated with a flow model is not discordant with the observations, its reaction torque on the local core fluid might
be diagnostic of failure of proper force balance in the core. Actually, this constraint seems to be strong; for the optimum flow to be obtained, it is necessary for flows to be still ambiguous enough to spare its unresolved space to meet the new constraint. We now examine the uncertainty of flow models in terms whether or not assuming the dominance of topographic conflict with the axial rotation of the mantle and core.

The principle of Hide’s method illustrates that, on decade timescales, the axial topographic torque contribute to the acceleration of the geostrophic flow \( \alpha^2 \), which belongs to the zeroth order flow \( \alpha^0 \) (Jault and Le Mouël, 1989). Here, \( \alpha = \sin \ beta \) denotes the radius of an axial cylinder in the spherical core, and \( \Sigma (s) \) is defined as the lateral surface of such a circular cylinder. The topographic torque \( \gamma (s) \) (in N) on a core annulus \( \Sigma (s) \) can be represented by (Jault and Le Mouël, 1989, 1990)

\[
\gamma (s) = \frac{\epsilon} {\sqrt{\epsilon^2 - \epsilon^0}} \int_0^\pi \int_0^{\frac{\pi}{2}} \left( h^+ \frac{\partial p}{\partial \theta} + h^- \frac{\partial p}{\partial \theta} \right) \sin \theta \, d\theta \, d\phi,
\]

where the superscripts \( \pm \) and \( \mp \) indicate equatorially symmetric and antisymmetric components at \( \tau = c \), respectively (the net axial topographic torque on the mantle in Eq. (11) is given by \( \Gamma_{cz} = - \int_0^c \gamma \, d\gamma \)).

The geostrophic flow is organized by the circulation of a slightly distorted annulus \( \Sigma (s) \) due to the CMB topology (Fearn and Proctor, 1992; Jault, 2003) which can be approximated at the order \( e \) by a circular annulus \( \Sigma (s) \) with the equal depth. Some algebra reveals that the topographic torque \( \gamma (s) \) given by Eq. (37) explains a part of the perturbation of the Taylor torque \( \gamma (s) \) on \( \Sigma (s) \) (\( \gamma (s) \approx \int \alpha \cdot T_F \, dS \)) with the body force \( F \) in Eq. (7)) that results from the distortion of the path of integration (Appendix A.2). Jault and Légaút (2005) remark that there still remains perturbation yielded due to the variable distance to the axis along the contour and the variable thickness of the annulus along the contour; the pressure torque acts even on the internal surface of the distorted annulus \( \Sigma (s) \). This is of the order of \( e \) (i.e. comparable to \( \gamma (s) \)) and cannot be evaluated with the observations. There can also be some possible torques unknown from observations (Zatman and Bloxham, 1999). In the present analysis, nevertheless, we try to assess the possible range of \( \gamma (s) \) calculated from the flow models, so that it can be compared with other torques. After all, the modified Taylor torque \( \gamma (s) \) on a distorted annulus \( \Sigma (s) \) may be written as

\[
\gamma (s) = \int \frac{(F \cdot dl) \, dz}{2 \pi (c, \phi)} = \gamma (s) + \gamma_1 (s) + \gamma_2 (s)
\]

(38)

where \( dl \) is the line element vector along the geostrophic contour and \( \gamma_1 (s) \) denotes uncalculated torques, such as the above-mentioned internal pressure torque. Time evolution of \( \Sigma (s) \) is then determined by

\[
\gamma (s) = \gamma (s)
\]

(39)

where \( \gamma (s) \approx 2 \pi R_0 \sqrt{\epsilon^2 - \epsilon^0} h \) \( \Sigma (s) \) is also approximated by \( \Sigma (s) \) in deriving \( \gamma (s) \).

The magnitude of the torque balance Eq. (39) may be inferred by roughly evaluating \( \gamma \) from decadal variation of \( \alpha^2 \) provided by a time series of core surface flow model. This turns out to be of order \( 10^{11} \) N.

In order to check if the flow realizes the state Eq. (40), we calculate \( \gamma (s) \) and its variance by using the estimated flows and their variances in Section 4. By applying Eq. (37), \( \gamma (s) \) can be represented as \( \Sigma (s) = \Sigma (s) \int \alpha \cdot T_F \, dS \) where \( \Sigma (s) \) includes associate Legendre function and \( h_\alpha \) and \( \chi_\alpha \) denote the coefficients \( \alpha \) and \( \beta \), respectively. Fig. 16 shows \( \gamma (s) \) related to the flow model A; it does not satisfy Eq. (40) and nor does any of those perturbed within its estimated standard deviation; the negative peak of \( \gamma (s) \), reaching \( -4 \times 10^{11} \) N near the equator (\( \varphi = 0.95 \) c, or \( \varphi = 18^\circ \) in latitude), is persistent. As a matter of fact, violated by the torques \( \gamma (s) \) indicates that the significant contribution arises from the torque related to the considerable flow \( \Sigma (s) \) interacting with the topography \( \delta \Sigma \), which hinders \( \gamma (s) \) from vanishing uniformly. This may imply that there is no way of constraining the flow by reducing \( \gamma (s) \) (Fig. 16).
true here that the state of γ typical spin-up timescale by the topographic coupling as quickly as several months, which is indicated by the Fig. 16. Axial topographic torque on coaxial cylinders of unit thickness in the core, Σ(s) (solid line in black) and its standard error (dashed line). Also plotted are Σ(s) associated with perturbed flows (gray lines).

Two possibilities may be invoked immediately for the interpretation of the above result. One is that the failure of Eq. (40) can be circumvented owing to imperfection lying in the present framework of calculating the topographic torque, and hence, the core is still in the state Eq.(40). The other is that Eq. (40) is not necessarily true, but the magnitude of γs is allowed to be larger than that of γF (−10^15 N) due possibly to γP or γF opposing it (Eq. (38)). The latter state may be unstable, however, because the energy supply is needed for the flow against the great dissipation if the opposing torque is of magnetic or frictional kind. Instead, the system would quickly alter the flow configuration to ensure Eq. (40) as is found by Kuang and Chao (2003) in the numerical experiment (note also the stabilizing mechanism suggested by Jault and Le Mouël (1993), in which density and pressure anomalies in the core are advected by the geostrophic flow uG which continuously adjust their distribution to keep γF from growing). This process might take place as quickly as several months, which is indicated by the typical spin-up timescale by the topographic coupling (εΣuG)^−1 (Jault, 2003). Thus, we will not think of it as true here that the state of γF is kept to have the magnitude of order 10^{15} N as in Fig. 16, though we cannot tell the significance of γP and other possible unknown torques.

The conflict between the calculated γF in Fig. 16 and the state Eq. (40) may be eliminated by the invalidity of the hypotheses of exact FF and TG. Incorporating magnetic diffusion or the Lorentz force at the CMB in the model might alter the flow field and pressure distribution, possibly reducing the magnitude of γF. However, it seems more plausible that the non-geostrophic belt lies mostly within the narrow band along the geographical equator (see Pais et al., 2004) and does not even stretch to the latitude corresponding to the negative peak of γF(s) in Fig. 16. With Hide’s method, a simple and prompt way to realize Eq. (40) would be to optimally modify the CMB topography just as demonstrated by Jault and Le Mouël (1990). Relative shift of the topography to the flow field by up to 2.5° (i.e. the resolution of the present topography model) about the Earth’s rotation axis makes some difference in γF(s), but is too ineffective to obtain Eq. (40); more optimized modification of the topography is necessary. The topography may have to be modified such that its amplitude is greatly reduced.

In the final analysis, it is found that reasonable constraint on TG flow uH in terms of the topographic coupling with the topography model by Boschi and Dziewonski (2000) is unavailable due to the insufficiency of the residual unconstrained space of uH. The torque equilibrium on the core annuli would be warranted most easily if the CMB topography is modified to establish the state Eq. (40). In order to obtain further insight concerning the dynamical role of the topographic torque γT(s), on timescales of decades or even shorter, it may be helpful to implement a time-dependent study of γT(s) and the torque equilibrium on the core cylinders Σ(s).

6. Conclusions

We have studied quantitatively the variability of the topographic core-mantle torque calculated by Hide’s method which relies on a TG core surface flow model. The various flow models are obtained which are compatible with the SV model of GSFC80 (Langel et al., 1982) and the MF model of DGRF1980. The variance of the flow models is also estimated by taking account only of that of the SV model as their source; the SV variance is scaled absolutely on the basis of the disagreement of the existing SV models around 1980. The subsequent topographic torques and their variance are then calculated with the CMB topography model by Boschi and Dziewonski (2000).

The magnitude of the calculated torque related to the TG flows with the reasonable rms velocity (∼15 km year^{-1}) is of the order of up to 10^{18} and 10^{19} Nm for the axial and equatorial components, respectively. Yet, the axial component can be of the same order of magnitude as 10^{17} Nm, which corresponds to the observed LOD variations. A still further ambiguity of the torque evaluation arises from the variance of flow models. This leads to the standard deviation of the torques somewhat larger than their typical magnitude, certifying the potential of the axial and equatorial torques to be as large as more than 10^{15} and 10^{16} Nm, respectively. The sensitiveness of these torques to the perturbation of the flows (within their standard deviation) is such that they
can be varied to that large extent in response to so small a variation of a flow model as hardly changing its rms velocity.

The net topographic torque calculated by Hide’s method is not affected significantly by smaller-scale interactions of the flow and the topography, as long as power spectrum of the flow decreases reasonably (like the flow model A in Section 4). The variability of the torque is ascribed mainly to that of the large-scale flows, or those interacting with the topography of degrees up to 6. Of particular contribution is the ambiguity of the flows related to the topography of degree 2 (including the flattening of the CMB) and 3; such flows effectively cause the torque variability on account both of their variation due to the non-uniqueness and of their variance due to that of the SV model. This is primarily because the invisible flow can be present even on large spatial scales owing to the morphology of the MF model. This large-scale invisible flow seems to comprise a major part of the notable regional flows such as the azimuthal flows along the equator (which are roughly along the contour of the quantity \(\psi = (B_r \cos \theta)\)).

Despite the variability of the calculated torque, no reasonable flow model explaining the observed decadal polar motion seems available. If the rms velocity of the flow is supposed to be \(-15\) km year\(^{-1}\), any of the predicted mean pole fails to reach the observed amplitude. As for the axial problem, the difficulty of the resultant torque lies rather in finding the consistency with the core dynamics than explaining the observed decadal LOD variations. With any of reasonable flows, the great topographic torques act on the annuli in the fluid core coaxial with the Earth’s rotation axis (larger by two orders of magnitude than the plausible inertial torque on such annuli). So far as Hide’s method is supposed to yield the estimate of the true torque, modification of the CMB model is necessary to achieve the appropriate Taylor state of the fluid core. Since the topography model is no less ambiguous than the flow model, it turns out after all that no further and effective constraint on the flow can be available besides the weak one for explaining the LOD variation.

We have sought possible range of the flow and concomitant topographic torque using the most pessimistic estimate of the SV variance. Recent continuous measurement of the geomagnetic field by satellite provides data associated with high spatial resolution throughout the globe, which greatly improves the quality of the SV model. They can reduce the variance of the model itself and also narrow the possible range of the flow spectrum (a decreasing trend close to \(n^{-2}\) is preferred as degree \(n\) increases (Eymin and Hulot, 2005), which lies between the requested trends by the norms \(N_2\) and \(N_4\) (Section 2.2)). With such a highly constrained flow model, the present argument would be a strict one, possibly leading to a stronger constraint on the CMB topography including smaller-scale structures.

The present study have been limited to the torque at a certain epoch 1980. However, time variation of the topographic torque should be investigated with the time series of geomagnetic and geodetic data (while the variance of earlier data become even larger), because temporal dependence of the core surface flow can be an additional constraint, and moreover because more detailed implications on the nature of core dynamics might be studied, such as the influence of the topographic coupling on the possible torque balance and its time variability on the annulus in the fluid core.

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Appendix A. The topographic torque in terms of the latitudinal core surface flow \(u_{z,0}\)

Following the concept of Hide’s method, one can derive the topographic torque in terms of the latitudinal component of the primary core surface flow \(u_{z,0}\), inferable from magnetic observations. This formalism does not include the pressure explicitly, so it may reflect a straightforward interpretation that the exchange of the angular momentum between the core and mantle is allowed by the CMB topography via secondary meridional circulation within the core caused by the compliance of primary core flow with the topography. This process is more effective by an order of \(10\) (Eq. (5)) to spin-up the core fluid than simple pressure action (like the wind pressure) where the Coriolis force is not important. In this appendix, the formalism is described using the spherical \((r, \theta, \psi)\) and cylindrical \((s, \psi, z)\) coordinates. The quantities of second order in \(e\) are all neglected. The asterisk (used in the main text to indicate dimensioned value) is dropped.

A.1. The fictitious Coriolis torque

It is confirmed here that the fictitious Coriolis torque \(\Gamma_0^*(u_{0})\) can be calculated from the topography and the latitudinal component of the core surface flow, and that
the axial topographic torque (on the core) \(- \Gamma_{x,z} \) coincides with \( \Gamma_{c}^{\nu}(u_{0}) \) if the TG (Eq. (10)) is valid at the core surface \((r = c) \).

A part of the axial component of the Coriolis torque (see Eqs. (3) for its definition) due only to the primary flow \( u_{0} \) (the fictitious Coriolis torque, Jault and Le Mouël, 1989) can be written as

\[
\Gamma_{c}^{\nu}(u_{0}) = \mathbb{I} \int_{V_{c}} \mathbf{r} \times (2 \Pi_{M} \mathbf{r} \times u_{0}) \, dV,
\]

where \( V_{c} \) is the volume of the core with bumpy surface, and \( \Sigma_{c}^{\nu} \) is the lateral surface of a circular annulus within the core. Using the incompressibility \( \nabla \cdot u_{0} = 0 \), one has

\[
\int_{\Sigma_{c}^{\nu}} u_{0} \, dS = \left[ \int_{\Sigma_{c}^{\nu} \cap \{ \theta = \pi \}} + \int_{\Sigma_{c}^{\nu} \cap \{ \theta < \pi \}} \right] \times g(\xi_{CMB}, \theta, \varphi) \, dS \tag{A.2}
\]

where \( \xi_{0} = \sin^{-1}(x/c) \) and \( g(\xi_{CMB}, \theta, \varphi) = -\hat{n} \cdot u_{0} \) is the apparent suction of \( u_{0} \) at the core surface (Eq. (13)) (see also Kuang and Chao, 2003, for the use of \( g \) in their numerical simulation of the topographic coupling). This suction can be written in terms of \( u_{0} \) at \( r = c \) (i.e. the core surface flow):

\[
g(\xi_{CMB}, \theta, \varphi) = -(r - \nabla \cdot h - \nabla \cdot h) \left( n_{\theta} + h \frac{\partial n_{\theta}}{\partial \varphi} \right) = \Omega \hat{\mathbf{r}} \cdot u_{0} + u_{0} \cdot \nabla \cdot h = \nabla \cdot (h u_{0})
\]

where \( u_{c}(\varphi) = 0 \) and \( \nabla \cdot u_{0} = 0 \) are used. \( \theta, \varphi, \zeta \)

Therefore, Eq. (A.2), given by the surface integral over the bumpy caps of the CMB at \( r = \xi_{CMB} \) inside the surface \( \Sigma_{c}^{\nu}(s) \), can be rewritten as the integral over the spherical caps at \( r = c \):

\[
\int_{\Sigma_{c}^{\nu}} u_{0} \, dS = 2\pi \int_{0}^{\pi} \int_{-\theta_{0}}^{\theta_{0}} \left( n_{\theta} + h \frac{\partial n_{\theta}}{\partial \varphi} \right) \sin \theta \, d\theta \, d\varphi
\]

\[
= \Omega \int_{0}^{\pi} \int_{-\theta_{0}}^{\theta_{0}} \hat{\mathbf{r}} \cdot u_{0} \sin \theta \, d\theta \, d\varphi \cdot d\varphi \tag{A.4}
\]

Substituting the above equation into Eq. (A.1), one finds the fictitious Coriolis torque in terms of the latitudinal component of the core surface flow \( u_{0} ; \theta \):

\[
\Gamma_{c}^{\nu}(u_{0}) = 4 \Pi_{M} \int_{0}^{\pi} \frac{\partial n_{\theta}}{\partial \varphi} \left( h \hat{n}_{\theta} \right) \left( h \hat{n}_{\theta} \right) \, d\varphi \tag{A.5}
\]

where \( \hat{\mathbf{r}} = \pm \sqrt{c^{2} - r^{2}} \) and \( h_{\pm, \theta} \) and \( u_{0}^{\pm, \theta} \) are the symmetric (+) and antisymmetric (−) part of \( h \) and \( u_{0} \) at \( z = \pm z_{0} \). This form of the fictitious Coriolis torque ensures that it can be calculated as long as one has models of the topography and core surface flow, no matter what kind of assumption is imposed in estimating the flow. If geomagnetic observations indicate the presence of latitudinal core surface flow \( u_{0} ; \theta \) (of zeroth order in \( c \)), there may well be some latitudinal flux that goes through the undulation of the CMB topography, generating the topographic torque of first order in \( c \). In Hide’s method where the flow model satisfies Eq. (10), the axial topographic torque on the mantle in Eq. (11)

\[
\Gamma_{x,x} = \int_{0}^{\pi} \int_{-\theta_{0}}^{\theta_{0}} \hat{\mathbf{r}} \cdot u_{0} \sin \theta \, d\theta \, d\varphi \tag{A.6}
\]

(which originates in its definition Eq. (1)) can be derived from the fictitious Coriolis torque \( \Gamma_{c}^{\nu}(u_{0}) \); one can readily find \(- \Gamma_{x,z} = \Gamma_{c}^{\nu}(u_{0}) \) by combining Eqs. (10) and (A.5).

A.2. Perturbation of the Taylor torque due to distorted annulus

It is shown in this subsection that Eq. (38) is valid to the first order in \( s \) (as long as the TG is true at \( r = c \), i.e. the topographic torque \( \gamma_{T}(s) \) in Eq. (37) corresponds to a part of perturbation of the Taylor torque \( \gamma_{T}(s) \) on a circular core annulus \( \Sigma(s) \), resulting from any body force acting along a distorted geostrophic annulus \( \Sigma_{c}(s) \) (of equal depth as \( \Sigma(s) \)) due to the CMB topography. This is an extension of the argument by Jault et al. (1996) in which the body force within the core is limited to the radial buoyancy (and so the TG is satisfied everywhere).

The body torque \( \gamma_{B}(s) \) on a distorted annulus can be written as

\[
\gamma_{B}(s) = \int_{\Sigma(s)} \mathbf{s} \cdot \mathbf{F} \cdot \nabla \cdot u \, dS = \gamma_{B}^{\nu}(s) + \gamma_{B}^{\theta}(s) + \gamma_{B}^{\varphi}(s)
\]

\[
\gamma_{B}^{\nu}(s) = \int_{\Sigma(s)} s F_{r} l_{r} \, dS \tag{A.7}
\]

\[
\gamma_{B}^{\varphi}(s) = \int_{\Sigma(s)} s F_{\varphi} l_{\varphi} \, dS \tag{A.8}
\]

\[
\gamma_{B}^{\theta}(s) = \int_{\Sigma(s)} s F_{\theta} l_{\theta} \, dS \tag{A.9}
\]

\[
\gamma_{B}^{\varphi}(s) = \int_{\Sigma(s)} s F_{\varphi} l_{\varphi} \, dS \tag{A.10}
\]

where \( s = (l_{r}, l_{\varphi}, l_{\theta}) \) is the unit vector parallel to the distorted geostrophic contour, and \( \mathbf{F} \) is the zeroth order body force within the core that emerges in Eq. (7) (see
also, Jault, 2003; Jault and Légaout, 2005). The distortion of $\Sigma(s)$ is represented by the displacement of its geostrophic contour in $s$ and $z$-direction at a certain longitude $\psi$ (denote them by $\delta h$ and $\delta z$). They can be given in terms of the equatorially symmetric and antisymmetric part of the topography $h(\theta, \psi)$:

$$\delta h(\psi) = \frac{c}{s} h^+(\theta_0, \psi), \quad \delta z(\psi) = \frac{c}{\sqrt{c^2 - s^2}} h^-(\theta_0, \psi).$$

(A.11)

Accordingly, the vector $F$ turns out to be

$$F_{\Sigma}(s, z) = \frac{1}{\sqrt{w_0^2 + z^2}} \frac{\partial}{\partial \theta} \rho \Omega_0 \mathbf{e}_z,$$

(A.12)

to the first order in $\epsilon$.

Now, one can express $\gamma_{0}^{(s)}$ in terms of the surface integral over the domain reduced to $\Sigma(s)$ (because of the small displacement) and separate it into $\gamma_{s}(s)$ and the perturbation:

$$\gamma_{0}^{(s)}(s) = \int_{\Sigma(s)} (\hat{r} + \delta h) F_\rho + \delta h \frac{\partial}{\partial \theta} F_\rho + \frac{\partial}{\partial \phi} F_\theta \ dS = \gamma_{s}(s) + \gamma_{0}(s)$$

$$+ \int_{\Sigma(s)} \left( \frac{ch^+}{s} \frac{\partial}{\partial \theta} F_\rho + \frac{ch^-}{s} \frac{\partial}{\partial \phi} F_\rho \right) \ dS$$

(A.13)

where $\gamma_{s}(s)$ is the perturbation written as

$$\gamma_{s}(s) = \int_{\Sigma(s)} \delta h F_\rho \ dS = 2\rho_0 \int_{\Sigma(s)} \delta h u_{0,0} \ dS$$

$$+ \frac{1}{\sqrt{w_0^2 + z^2}} \delta z u_{0,0} \ dS \tag{A.14}$$

which is found by $\psi$-component of Eq. (7) to be generated by the Coriolis force and the pressure acting on the internal surface of the distorted annulus (see also Jault and Légaout, 2005). $\gamma_{0}(s)$ and $\gamma_{0}^{(s)}$ can be written as

$$\gamma_{0}(s) = \int_{\Sigma(s)} \frac{c}{\sqrt{c^2 - s^2}} \frac{\partial}{\partial \phi} F_\theta \ dS = - \int_{\Sigma(s)} \frac{ch^-}{s} \frac{\partial}{\partial \phi} F_\theta \ dS \tag{A.15}$$

and

$$\gamma_{0}^{(s)} = \int_{\Sigma(s)} \frac{c}{\sqrt{c^2 - s^2}} \frac{\partial}{\partial \phi} F_\theta \ dS = - \int_{\Sigma(s)} \frac{ch^-}{s} \frac{\partial}{\partial \phi} F_\theta \ dS \tag{A.16}$$

(note that domain of the integral is also reduced to $\Sigma(s)$ in the above equations). By substituting the $z$- and $s$-component of $\nabla \times \mathbf{F}$ into Eq. (A.15) and (A.16), respectively, one has

$$\gamma_{0}(s) = - \int_{\Sigma(s)} \frac{ch^-}{s} \left( 2\rho_0 \Omega_0 \frac{\partial u_{0,0}}{\partial \theta} - \frac{\partial}{\partial \phi} F_\theta \right) \ dS \tag{A.17}$$

$$\gamma_{0}^{(s)} = - \int_{\Sigma(s)} \frac{ch^-}{s} \left( \frac{\partial}{\partial \phi} F_\theta - 2\rho_0 \Omega_0 h^- \frac{\partial u_{0,0}}{\partial \phi} - \frac{\partial}{\partial \phi} F_\theta \right) \ dS \tag{A.18}$$

From Eqs. (A.13), (A.17) and (A.18), Eq. (A.7) proves to be

$$\gamma_{0}(s) = \gamma_{s}(s) + \gamma_{0}(s)$$

(A.19)

where $\gamma_{C}(s)$ is the fictitious Coriolis torque in Eq. (A.5) (the primary flow satisfies $u_{0,0} = -(s/c)u^*_{0,0}$ and $u^*_{0,0} = (\sqrt{c^2 - s^2}/c)u^*_{0,0}$ at $r = c$ because of the boundary condition $\mathbf{u}(r = c) = 0$). A part of the perturbation of the Taylor torque $\gamma_{s}(s)$ due to the body force exerted along the deflected path of integration coincides with $\gamma_{C}(s)$ (which can be evaluated directly from models of the topography and the latitudinal core surface flow).

The $\psi$-component of Eq. (7) verifies the identity

$$\gamma_{0}(s) = \gamma_{s}(s) + \gamma_{0}(s),$$

where $\gamma_{C}(s)$ is the topographic torque (Eq. (37)) and $\gamma_{0}(s)$ is the torque written as

$$\gamma_{0}(s) = \int \frac{h^+ u_{0}^*}{s} + \frac{h^- u_{0}^*}{\sqrt{c^2 - s^2}} \ d\psi \tag{A.20}$$

which arises from the body force $\mathbf{F}$ acting on the undulation of the topography ($\gamma_{s}^{T} = - \int \gamma_{s} F_{\rho} \ dS = - \int \gamma_{s} F_{\rho} (\hat{r} + \delta h) F_\rho \ dS$), corresponding to the perturbation of the gravitational and electromagnetic core-mantle torque due to the introduction of a non-axisymmetric CMB topography. Eq. (38) can still be valid in Hidé’s method where the TG is assumed at $r = c$, because it makes $\gamma_{s}^{T}$ or $\gamma_{0}^{T}$, attributed to the normal surface force at the boundary (such as the magnetotopographic torque resulting from the magnetic pressure (Kuang and Bloxham, 1997)) and hence $\gamma_{C}$ can be redefined to incorporate $\gamma_{C}$ in itself. In any case (with or without the complete TG throughout the core surface), $\gamma_{C}$ should be much smaller than $\gamma_{0}$ in reality, unless an almost complete cancellation with the azimuthal integral in $\gamma_{C}$ takes place.
References


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