Geophys. J. Int. (1998) **134,** 849–855

Normal-mode solutions for radiation boundary conditions with an impedance contrast

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Accepted 1998 April 20. Received 1998 April 17; in original form 1998 January 8

SUMMARY

Wave propagation problems with radiation boundaries cannot be solved by the ordinary eigenfunction expansion method because not all of the eigenfunctions are mutually orthogonal due to non-Hermitian boundary conditions. We present a method for solving such problems in terms of a superposition of eigenfunctions, using the biorthogonal eigenfunction expansion method outlined by Morse & Feshbach (1953). We develop their method, using a variational equation, so that the calculations other than those of eigenfunctions are unnecessary to construct the solution when there is an impedance contrast at the radiation boundary. We present numerical computations for a 1-D semi-infinite continuum that has an impedance contrast. This method may be applicable to such problems as the vibration of a magma chamber embedded in the crust, the acoustic coupling between the solid Earth and the atmosphere, and wave propagation in a layered half-space.

Key words: elastic wave theory, layered media, normal modes, synthetic seismograms, wave propagation.

1 INTRODUCTION

The excitation problems with radiation boundary conditions that have an impedance contrast (for example, a magma chamber embedded in the crust, a vertically layered half-space, and the solid Earth in the atmosphere) have not usually been considered by the normal-mode approach. The reason may be that the normal modes are no longer 'normal', or orthogonal, due to non-Hermitian radiation boundary conditions and therefore the ordinary modal approach cannot be used. Fujita, Ida & Oikawa (1995) and Sakuraba, Imanishi & Oikawa (1995) studied the free oscillation of a fluid sphere embedded in an infinite elastic medium to understand the source mechanism of volcanic tremors. They investigated the response of the system to an applied force using a Green's function technique, but they treated only peculiar sources with spherical symmetry due to the intricacy of analytical calculation. The vertically layered half-space problem has been studied by various investigators (e.g. Ewing, Jardetzky & Press 1957). It has usually been treated by a Green's function technique such as the reflectivity method (e.g. Kennett 1983). The total wavefield, including body waves, cannot be represented by an orthogonal set of modes whose energy is trapped in the structure. To complement the part of the wavefield, whose energy leaks into the underlying half-space, Haddon (1987) considered the contribution of leaking modes. However, he did not use the ordinary eigenfunction expansion method. Maupin (1996) used the radiation modes, which are obtained by subjecting a weaker boundedness condition at depth than ordinary radiation boundary conditions. In the study of the coupled oscillation of the atmosphere and the Earth excited by volcanic eruptions, the effect of the modes whose energy radiates outwards is not calculated because the treatment of non-Hermitian boundary conditions is difficult (Watada 1995).

It is, however, possible to solve problems involving non-Hermitian differential operators entirely in terms of eigenfunctions by considering both the original non-Hermitian system and its Hermitian adjoint (Morse & Feshbach 1953, henceforth M&F). The original eigenfunctions and their Hermitian adjoint eigenfunctions are orthogonal (biorthogonality), thus we can solve the excitation problem.

In this paper, we outline the application of M&F's biorthogonal eigenfunction expansion method to a problem that can also be solved in a straightforward manner by conventional means: a 1-D string with a radiation boundary that has an impedance contrast. We choose this problem because it includes all of the essential features of the aforementioned problems.

2 BIORTHOGONAL EIGENFUNCTIONS

We consider the solution of the inhomogeneous equation

$$(\hat{L}-\nu\hat{\rho})w=f \tag{1}$$

using eigenfunction expansion methods, where $\hat{L}$ and $\hat{\rho}$ represent the linear differential operators (including the boundary conditions), ν is an arbitrary number and f is the inhomogeneous term. The inner product of the two functions

a and b is defined as

$$(a \cdot b) \equiv \int_1 a^*(x) b(x)\, dx, \tag{2}$$

where * denotes the complex conjugate and dx may represent an integral over either one or several dimensions. If $\hat{L}$ and $\hat{\rho}$ are both Hermitian, the method of solution is well known. We solve the eigenvalue equation

$$\hat{L} V_n = \lambda_n \hat{\rho} V_n. \tag{3}$$

Since $\hat{L}$ and $\hat{\rho}$ are Hermitian, it can be proved that the eigenfunctions $\{V_n\}$ are mutually orthogonal with respect to $\hat{\rho}$:

$$(V_m \cdot \hat{\rho} V_n) = 0 \quad \text{if} \quad n \neq m. \tag{4}$$

We solve the system (1) by expanding w in terms of $\{V_n\}$ and using the orthogonality (4). We obtain

$$w = \sum_n \frac{(V_n \cdot f)}{(\lambda_n - v)(V_n \cdot \hat{\rho} V_n)} V_n. \tag{5}$$

However, if $\hat{L}$ is non-Hermitian and $\hat{\rho}$ is Hermitian, the eigenfunctions are not necessarily orthogonal and thus we cannot, in general, use the standard eigenfunction expansion techniques to find a solution. Using the biorthogonal eigenfunction expansion method of M&F, we solve the eigenvalue equations for the original system and the adjoint system:

$$\hat{L} V_n = \lambda_n \hat{\rho} V_n, \tag{6}$$

$$\hat{L}^* U_m = \mu_m \hat{\rho} U_m, \tag{7}$$

where $\hat{\ }^*$ denotes the Hermitian conjugate operator. These eigenfunctions and eigenvalues have the following properties:

$$\mu_n^* = \lambda_n, \tag{8}$$

$$(U_m \cdot \hat{\rho} V_n) = 0 \quad \text{if} \quad n \neq m \tag{9}$$

(e.g. M&F). When $\hat{L}$ is Hermitian, i.e. $\hat{L}$ is equal to $\hat{L}^*$, $V_n = U_n$ and $\lambda_n = \mu_n$. Thus (8) indicates that the eigenvalues are real and (9) indicates the orthogonality of the eigenfunctions. Note that when $\hat{L}$ is not Hermitian the set of eigenfunctions $\{V_n\}$ is not necessarily mutually orthogonal but the functions $\{U_n\}$ have orthogonal relationships with $\{V_n\}$. This relationship is called *biorthogonality*.

We solve the system (1) by expanding w in terms of $\{V_n\}$ and using the biorthogonality (9). We obtain

$$w = \sum_n \frac{(U_n \cdot f)}{(\lambda_n - v)(U_n \cdot \hat{\rho} V_n)} V_n. \tag{10}$$

When $\hat{L}$ is Hermitian this method reduces to the familiar eigenfunction expansion method.

3 1-D EXAMPLE

We consider a 1-D semi-infinite 'string', with variable stiffness A and density ρ. We set

$$A(x) = \begin{cases} A_1 & 0 \le x \le l \\ A_2 & x \ge l \end{cases},$$

$$\rho(x) = \begin{cases} \rho_1 & 0 \le x \le l \\ \rho_2 & x \ge l \end{cases},$$

$$Z \equiv \frac{\rho_2 c_2}{\rho_1 c_1} = \frac{Z_2}{Z_1}, \tag{11}$$

where Z_i is the impedance, Z is the impedance contrast, $c_1 = \sqrt{A_1/\rho_1}$ and $c_2 = \sqrt{A_2/c_2}$. We assume $Z \neq 1$. We fix the string at $x = 0$. At $x = l$ we require that waves radiate outwards and do not reflect from the end. Displacement and traction are continuous across $x = l$. We consider the problem of finding the displacement of the string $w(x, t)$ when a unit impulse is given at the point $x = \xi (0 \le \xi \le l)$ at $t = 0$ (Fig. 1).

The Fourier transformed equation of motion is

$$\left[\frac{d}{dx}\left(A(x)\frac{d}{dx}\right) + \omega^2 \rho(x)\right] w(x, \omega) = -\delta(x - \xi), \tag{12}$$

with boundary conditions

$$w(0, \omega) = 0 \qquad \text{fixed at } x = 0, \tag{13}$$

$$w^+(l, \omega) - w^-(l, \omega) = 0$$

$$\text{continuity of displacement across } x = l, \tag{14}$$

$$A_2 \frac{d}{dx} w^+(l, \omega) - A_1 \frac{d}{dx} w^-(l, \omega) = 0$$

$$\text{continuity of traction across } x = l, \tag{15}$$

$$\frac{d}{dx} w^+(l, \omega) - i \frac{\omega}{c_2} w^+(l, \omega) = 0$$

$$\text{radiation boundary}, \tag{16}$$

where ω is the frequency.

The eigenfunctions of this system are

$$v_n(x) = \begin{cases} \sin(\omega_n x/c_1) & 0 \le x \le l \\ \sin(\omega_n l/c_1)\, e^{i(\omega_n/c_2)(x-l)} & x \ge l \end{cases}, \tag{17}$$

where ω_n is the nth root of the equation

$$\omega = \frac{1}{Z} \omega \frac{e^{i\omega l/c_1} + e^{-i\omega l/c_1}}{e^{i\omega l/c_1} - e^{-i\omega l/c_1}}, \tag{18}$$

chosen so that $\mathcal{R}e\, \omega_n \ge 0$. We can solve (18) analytically to obtain

$$\omega_n = \begin{cases} \dfrac{c_1}{l}(n\pi - i \coth^{-1} Z) & \text{for } Z > 1 \\ \dfrac{c_1}{l}\left(\left(n + \dfrac{1}{2}\right)\pi - i \tanh^{-1} Z\right) & \text{for } Z < 1 \end{cases} \quad (n = 0, 1, 2, \ldots). \tag{19}$$

Fig. 2 shows the eigenfunctions when $Z \gtrless 1$.

The goal of this paper is to set up the transient solution $w(x, t)$ in terms of the eigenfunctions $\{v_n\}$. However, it is

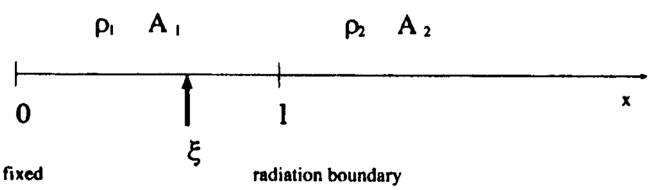

Figure 1. The semi-infinite string model showing source and boundaries.

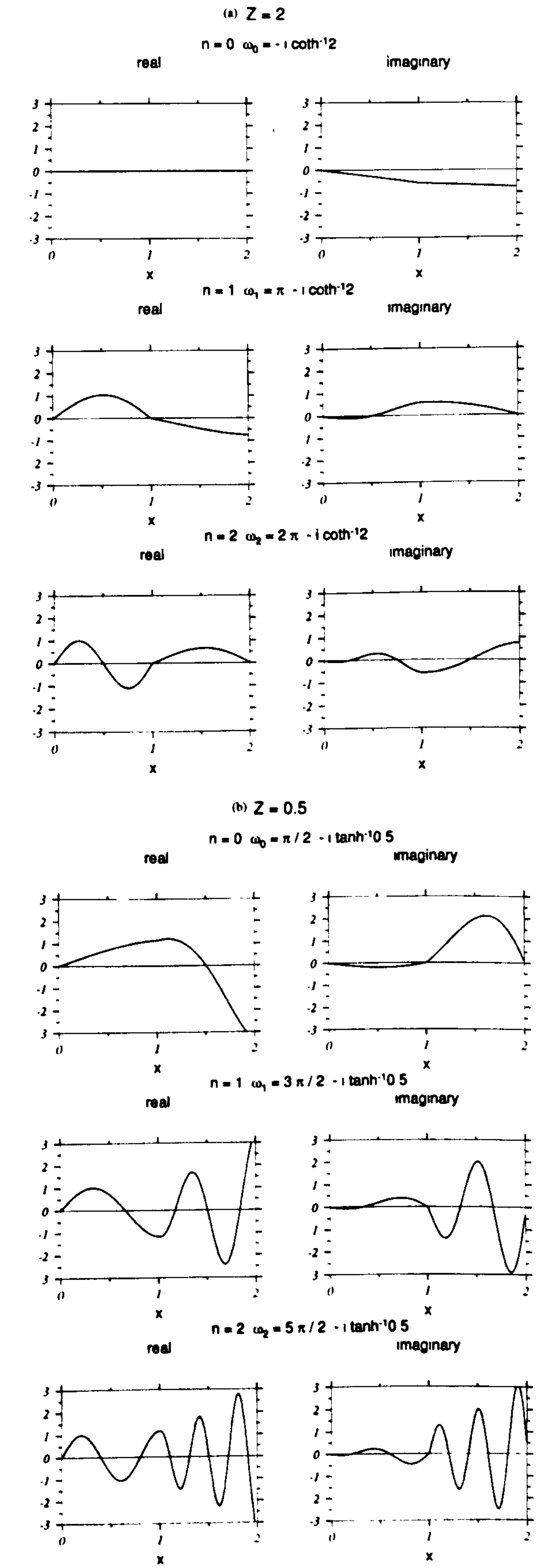

Figure 2. (a) The eigenfunctions for modes $n=0$ to 2. $c_1=1$, $Z=2$, $l=1$. (b) The eigenfunctions for modes $n=0$ to 2. $c_1=1$, $Z=0.5$, $l=1$.

inconvenient to use $\{v_n\}$ as the basis to represent the inhomogeneous solution $w(x, \omega)$ in the frequency domain . The reason is that v_n does not satisfy the boundary condition required for $w(x, \omega)$ at $x=l$: v_n satisfies

$$\frac{d}{dx}\, v_n^+(l) - i\,\frac{\omega_n}{c_2}\, v_n^+(l) = 0\,, \tag{20}$$

while $w(x, \omega)$ is subjected to (16). When we determined ω_n and v_n, we solved the homogeneous equation (12) with (13)–(16), considering that the term ω, which appears in (12) and (16), is *common* and corresponds to ω_n. As a result, the boundary condition for v_n at $x=l$ depends on the eigenfrequency ω_n. Therefore, it is difficult to expand $w(x, \omega)$ in terms of v_n, each of which satisfies its own boundary condition (20). In the following section, we use the basis that satisfies (16).

We separate the problem into two parts: the inside $(0 \le x \le l)$ and the outside $(x \ge l)$ of the radiation boundary. First we consider the motion of the inner string, and then extend it to the outer region using (14) and (15).

3.1 The inner string

The linear operators that appear on the left-hand side of (12) are formally Hermitian. We can define the boundary conditions of the inner string completely as

$$V(0, \omega) = 0\,, \tag{21}$$

$$A_1\,\frac{d}{dx}\,V(l, \omega) - i\omega Z_2 V(l, \omega) = 0 \quad \text{radiation boundary}\,, \tag{22}$$

where V is the displacement of this system. (22) is obtained by inserting the continuity conditions (14) and (15) into (16). The Hermitian adjoint boundary conditions, which can be derived using Green's identity (e.g. M&F), are

$$U(0, \omega) = 0\,, \tag{23}$$

$$A_1\,\frac{d}{dx}\,U(l, \omega) + i\omega^* Z_2 U(l, \omega) = 0\,, \tag{24}$$

where U is the displacement of the Hermitian adjoint system. The radiation boundary condition is non-Hermitian because (22) and (24) differ. (24) is equivalent to an incoming radiation boundary condition.

3.1.1 Frequency-domain solution

Here we shall follow M&F in deriving a complete representation of the total wavefield as a sum of eigenfunctions. We introduce the basis $\{V_{n_\omega}\}$ that satisfies

$$\left(A_1\,\frac{d^2}{dx^2} + \rho_1 \Omega_{n_\omega}{}^2\right) V_{n_\omega}(x, \omega) = 0 \tag{25}$$

with boundary conditions (21) and (22). We obtain

$$V_{n_\omega}(x, \omega) = \sin\left(\Omega_{n_\omega}(\omega) x / c_1\right), \tag{26}$$

where $\Omega_{n_\omega}(\omega)$ is the n_ωth root of the transcendental equation

$$\omega = \frac{1}{Z}\,\Omega\,\frac{e^{i\Omega l/c_1} + e^{-i\Omega l/c_1}}{e^{i\Omega l/c_1} - e^{-i\Omega l/c_1}}\,, \tag{27}$$

chosen so that $\mathcal{Re}\,\Omega_{n_\omega} \ge 0$.

Note the difference between V_{n_ω} and v_n. We determine V_{n_ω} by considering ω as a constant coefficient of $iZ_2 V(l, \omega)$ of the radiation boundary condition (22), while we determine v_n by considering that ω, which appears in the radiation boundary condition, is identical to the eigenfrequency. V_{n_ω} satisfies the boundary condition (22) required for the inhomogeneous solution (22 is essentially the same as 16). Thus in contrast to $\{v_n\}$, $\{V_{n_\omega}\}$ is a suitable basis to represent the inhomogeneous solution in the frequency domain. Ω_{n_ω} and V_{n_ω} are functions of ω. When $\omega=\omega_n$, V_{n_ω} corresponds to v_n and $\Omega_{n_\omega}(\omega)$ corresponds to ω_n [set $\omega=\Omega$ in (27) and compare it to (18)].

As we showed in an earlier section, this is a non-Hermitian problem. In order to obtain the expansion coefficient, we introduce the Hermitian adjoint basis $\{U_{n_\omega}\}$, which satisfies

$$\left(A_1\frac{d^2}{dx^2}+\rho_1\Omega_{n_\omega}^{*\,2}\right)U_{n_\omega}(x,\omega)=0 \tag{28}$$

with boundary conditions (23) and (24). We obtain

$$U_{n_\omega}(x,\omega)=\sin\left(\Omega_{n_\omega}^*(\omega)x/c_1\right). \tag{29}$$

When $\omega=\omega_n$, U_{n_ω} corresponds to u_n, which is the eigenfunction of the Hermitian adjoint system. (We can obtain $\{u_n\}$ by following the same procedure as $\{v_n\}$, but with an incoming radiation boundary condition. In this case u_n becomes the complex conjugate of v_n.) We can now set $\nu=-\omega^2$, $\lambda_n=-\Omega_{n_\omega}^2$, $\hat\rho=\rho_1$ and $f=-\delta(x-\xi)$ in (10) and use the general result of the biorthogonal eigenfunction expansion to obtain the motion of the string:

$$w(x,\omega)=\sum_{n_\omega=0}^{\infty}\frac{-U_{n_\omega}^*(\xi,\omega)V_{n_\omega}(x,\omega)}{(\omega^2-\Omega_{n_\omega}^2)(U_{n_\omega}\cdot\rho_1V_{n_\omega})}. \tag{30}$$

Although this result takes a similar form to the ordinary modal solution, it is laborious to obtain the transient solution in the frequency domain. There are infinite discrete eigenstates for each ω. To calculate (30) we have to solve (27) and sum the eigenstates for each ω.

3.1.2 Time-domain solution

The displacement as a function of time can be found by inverting the Fourier transform

$$w(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{n_\omega=0}^{\infty}\frac{-U_{n_\omega}^*(\xi,\omega)V_{n_\omega}(x,\omega)}{(\omega^2-\Omega_{n_\omega}^2)(U_{n_\omega}\cdot\rho_1V_{n_\omega})}e^{-i\omega t}d\omega \tag{31}$$

or, interchanging integration and summation,

$$w(x,t)=\sum_{n_\omega=0}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{-U_{n_\omega}^*(\xi,\omega)V_{n_\omega}(x,\omega)}{(\omega^2-\Omega_{n_\omega}^2)(U_{n_\omega}\cdot\rho_1V_{n_\omega})}e^{-i\omega t}d\omega\right]. \tag{32}$$

This integrand has simple poles at the solutions of the equations

$$\omega=\pm\Omega_{n_\omega}(\omega), \tag{33}$$

which reduce to the secular equation of the original system and its Hermitian adjoint system; for each value of n_ω in Ω_{n_ω} there is a pair of roots, $\omega=\omega_n$, $-\omega_n^*$ [we defined the index n_ω so that the roots of (33) correspond to ω_n and $-\omega_n^*$]. These poles always lie in the lower half-plane.

For $t<0$ we use the ω-axis upper-semi-circle contour, and there are no poles inside. Thus the integral is zero. For $t>0$ we use the ω-axis lower-semi-circle contour (e.g. Aki & Richards 1980). We have

$$\int_{-\infty}^{\infty}\frac{-U_{n_\omega}^*(\xi,\omega)V_{n_\omega}(x,\omega)}{(\omega^2-\Omega_{n_\omega}^2)(U_{n_\omega}\cdot\rho_1V_{n_\omega})}e^{-i\omega t}d\omega$$

$$=2\pi i\left[\mathrm{Res}_{\omega=\omega_n}+\mathrm{Res}_{\omega=-\omega_n^*}\right], \tag{34}$$

where Res denotes the residue of the integrand at its poles, and

$$\mathrm{Res}_{\omega=\omega_n}$$

$$=\frac{-U_{n_\omega}^*(\xi,\omega_n)V_{n_\omega}(x,\omega_n)}{\frac{\partial}{\partial\omega}(\omega-\Omega_{n_\omega})|_{\omega=\omega_n}(\omega_n+\Omega_{n_\omega}(\omega_n))(U_{n_\omega}(x,\omega_n)\cdot\rho_1V_{n_\omega}(x,\omega_n))}$$

$$\times e^{-i\omega_nt}$$

$$=\frac{-u_n^*(\xi)v_n(x)}{\left(1-\frac{\partial\Omega_n}{\partial\omega}\Big|_{\omega=\omega_n}\right)2\omega_n(u_n\cdot\rho_1v_n)}e^{-i\omega_nt}, \tag{35}$$

where $\Omega_{n_\omega}(\omega_n)=\omega_n$, $V_{n_\omega}(x,\omega_n)=v_n(x)$ and $U_{n_\omega}(x,\omega_n)=u_n(x)$ were used.

Calculation of $\partial\Omega_{n_\omega}/\partial\omega|_{\omega=\omega_n}$ is performed with the aid of the variational equation (e.g. Aki & Richards 1980). The variational equation for this string is

$$\delta I_1-\delta B-\Omega^2\delta I_2=0, \tag{36}$$

where

$$I_1=\frac{1}{2}\int_0^l A_1\left(\frac{dV}{dx}\right)^2dx, \tag{37}$$

$$I_2=\frac{1}{2}\int_0^l\rho_1V^2(x)dx, \tag{38}$$

$$B=\frac{i}{2}\omega Z_2V^2(l), \tag{39}$$

$$\Omega^2=\frac{I_1-B}{I_2}. \tag{40}$$

Differentiating (40) with respect to ω at the stationary point and using (36), we finally obtain the derivative of the eigenfrequency

$$\frac{\partial\Omega_{n_\omega}}{\partial\omega}=-\frac{iZ_2V_{n_\omega}^2(l,\omega)}{4\Omega_{n_\omega}I_2}. \tag{41}$$

Thus at $\omega=\omega_n$ we have

$$\frac{\partial\Omega_{n_\omega}}{\partial\omega}\Big|_{\omega=\omega_n}=-\frac{iZ_2v_n^2(l)}{4\omega_nI_2}. \tag{42}$$

[Of course, in this simple example we can obtain (42) directly by differentiating (27).] We can obtain the residues at $\omega=-\omega_n^*$ by following the same procedure.

Computing the residues about each pole, we see that

$$w(x,t)=\begin{cases}0 & \text{for } t<0,\\ \sum_n \mathscr{Im}\left[\frac{v_n(\xi)v_n(x)e^{-i\omega_nt}}{\omega_n(v_n^*\cdot\rho_1v_n)+iZ_2v_n^2(l)/2}\right] & \text{for } t>0,\end{cases} \tag{43}$$

where $u_n^* = v_n$ was used. (43) can be written explicitly as

$$w(x, t) = \begin{cases} \sum_{n=1}^{\infty} \mathcal{Im}\left[\frac{2\sin(\omega_n\xi/c_1)\sin(\omega_n x/c_1)\,e^{-i\omega_n t}}{\rho_1 l\omega_n}\right] \\ \quad -\frac{i\sin(\omega_0\xi/c_1)\sin(\omega_0 x/c_1)\,e^{-i\omega_0 t}}{\rho_1 l\omega_0} & \text{for } Z > 1, \\ \sum_{n=0}^{\infty} \mathcal{Im}\left[\frac{2\sin(\omega_n\xi/c_1)\sin(\omega_n x/c_1)\,e^{-i\omega_n t}}{\rho_1 l\omega_n}\right] & \text{for } Z < 1. \end{cases} \quad (44)$$

The final solution is expressed in terms of the true eigenfunctions $\{v_n\}$, which are not mutually orthogonal. We started with the biorthogonal set of eigenfunctions, $\{V_{n_\omega}\}$ and $\{U_{n_\omega}\}$, which are functions of ω. However, in fact we do not have to calculate them explicitly because the inverse Fourier transformed solution is given by the residue contributions from the poles, which correspond to the eigenfrequency of the original and Hermitian adjoint systems. Usually, the time-domain solution can be expressed in terms of the eigenfunctions of the original and Hermitian adjoint systems. In this example the Hermitian adjoint eigenfunction is the complex conjugate of the original one and therefore we can represent $w(x, t)$ as a sum of $\{v_n\}$. Thus the response of the system to an arbitrary source can be obtained once the true eigenfunctions have been calculated.

3.2 The outer string

Next we consider $x \geq l$. We extend the solution of the inner string using the continuity of displacement and traction. We find that the solution of the outer string can also be expressed in terms of the true eigenfunction $\{v_n\}$ as (43). It can be written explicitly as

$$w(x, t) = \begin{cases} \sum_{n=1}^{\infty} \mathcal{Im}\left[\frac{2\sin(\omega_n\xi/c_1)\sin(\omega_n l/c_1)\,e^{i\omega_n\{(x-l)/c_2 - t\}}}{\rho_1 l\omega_n}\right] \\ \quad -\frac{i\sin(\omega_0\xi/c_1)\sin(\omega_0 l/c_1)\,e^{i\omega_0\{(x-l)/c_2 - t\}}}{\rho_1 l\omega_0} & \text{for } Z > 1, \\ \sum_{n=0}^{\infty} \mathcal{Im}\left[\frac{2\sin(\omega_n\xi/c_1)\sin(\omega_n l/c_1)\,e^{i\omega_n\{(x-l)/c_2 - t\}}}{\rho_1 l\omega_n}\right] & \text{for } Z < 1. \end{cases} \quad (45)$$

3.3 Green's function

The Green's function for this system can be found straightforwardly by finding the response of the system to an impulse at $x = \xi$ and $t = 0$. We break the Green's function into two parts, for $x < \xi$ and $x > \xi$, and use the continuity of w and the jump discontinuity of dw/dx at the point of the impulse. We apply the boundary condition by requiring the waves to travel only to the right for $x > l$. The Green's function is

$$w(x, \omega) = \begin{cases} \frac{1}{\omega Z_1}\frac{(Z-1)\,e^{ik_1(l-x_>)} - (Z+1)\,e^{-ik_1(l-x_>)}}{(Z-1)\,e^{ik_1 l} - (Z+1)\,e^{-ik_1 l}}\sin(k_1 x_<) & 0 \leq x \leq l, \\ -\frac{2}{\omega Z_1}\frac{\sin(k_1\xi)}{(Z-1)\,e^{ik_1 l} - (Z+1)\,e^{-ik_1 l}}\,e^{ik_2(x-l)} & x \geq l, \end{cases} \quad (46)$$

where $k_i = \omega/c_i$. $x_<$ and $x_>$ denote the smaller and larger of the two variables x and ξ.

Eqs (30) and (46) provide two very different representations for $w(x, \omega)$; however, we can reconcile them easily in the time domain using the fact that (46) has simple poles at $\omega = \pm\omega_n$. Thus the two representations are wholly equivalent.

4 NUMERICAL EXAMPLE

We present a numerical example of the biorthogonal expansion method. We set $A_1 = 1$, $A_2 = 2$, $\rho_1 = 1$, $\rho_2 = 1$ and $l = 1$. Thus $c_1 = 1$ and $c_2 = \sqrt{2}$. Instead of $\delta(t)$ time dependence, the time dependence of (44) and (45) is $(d/dt)(\delta(t)) * \exp(-t^2/\tau_0^2)$ (where τ_0 is the constant 'rise time'), and we get basically Gaussian pulses travelling along the string. We calculated the excitation for the solution by the biorthogonal eigenfunction expansion method for the case of $\xi = 0.5$ and $\tau_0 = 0.001$. Fig. 3 shows snapshots of the displacement along the string for times

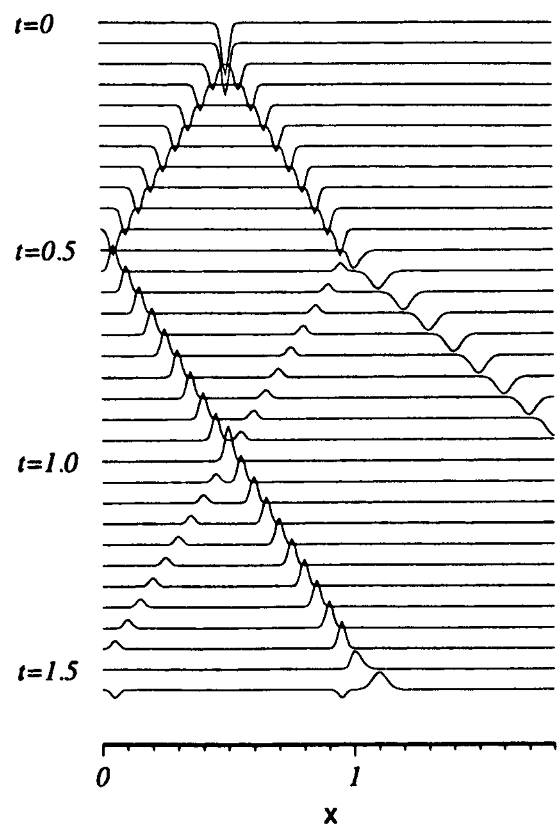

Figure 3. 'Snapshots' of the string at time intervals of $\Delta t = 0.05$. 60 modes are summed.

from $t=0$ to $t=1.55$. In each calculation, 60 modes are used. From $t=0$ to $t=0.5$ the pulses propagate in both directions away from the source. The effect of the boundary occurs at $t=0.55$. The pulse travelling on the left has been reflected by the rigid boundary and now travels to the right (away from the boundary) with the negative of its previous amplitude. Meanwhile, the rightmost pulse has passed through the discontinuity at $x=l$. The reflection and transmission coefficients for the wave travelling to the right are

$$R=\frac{1-Z}{1+Z}, \quad (47)$$

$$T=\frac{2}{1+Z}. \quad (48)$$

For our problem, $R=-1/3$, $T=2/3$. These coefficients for the biorthogonal eigenfunction expansion solution agree with the theoretical values.

5 DISCUSSION AND CONCLUSIONS

We have solved a simple problem with non-Hermitian, radiating boundary conditions with an impedance contrast using the biorthogonal eigenfunction expansion method of M&F. Our 1-D example is essentially the same as the example that M&F presented: a string of length l fixed at $x=0$ and supported non-rigidly at $x=l$, where the non-rigid support applies the resistive force on the string and the force is proportional to the velocity. However, they overlooked the fact that Ω_n is a function of ω in the calculation of the residues. Thus their result is incorrect (M&F eq. 11.1.29, p. 1347), although their formulation is correct. We have presented here the correct calculation of the residues using the variational principle.

Our time-domain method cannot be used when there is no impedance contrast at the radiation boundary (i.e. $Z=1$), since there is no non-trivial eigensolution of the free oscillation. It is extremely laborious to obtain the solution by the eigenfunction expansion method in that case: we have to solve the secular equation (27), sum the eigenstates for each ω, and then inverse Fourier transform back to the time domain. Geller, Noack & Fetter (1985) solved the excitation problem of a 1-D string with a radiation boundary that has no impedance contrast using the shifted eigenvalue method of Lanczos (1961). Their method is very computationally intensive and is thus of practically no use.

We determined the solution inside the radiation boundary first, and then extended it to the outer region. Our procedure may seem artificial because the usual limit in the integral in mode methods of a vertically layered half-space is ∞, although the radiation boundary is set at a finite depth. The boundary condition at infinite depth for surface waves is no displacement, since the energy is trapped in the structure. In our problem the energy leaks into the half-space and therefore the displacement of modes diverges as $x\rightarrow\infty$. We cannot properly define the boundary condition at $x=\infty$, but we can fully express the effect of the outer string by the boundary condition for the inner string at $x=l$ (22). Using the continuity of displacement and traction, the solution of the outer string is determined uniquely once the solution of the inner string has been determined. Thus we separate the problem into two parts, and do not extend the limits in the integral of the inner product and the energy integrals (37) and (38) to infinity.

Although we have presented this method for a 1-D case, it can easily be extended to 2- or 3-D problems of laterally homogeneous media with cylindrical, Cartesian or spherical coordinates, because after some integral transformation, these problems reduce to the same form as (12). This method may make useful contributions to such problems as the vibration of a magma chamber embedded in the crust, wave propagation in a layered half-space, and the acoustic coupling between the solid Earth and the atmosphere.

For example, in previous studies of volcanic tremors, the vibration of a magma chamber excited inside the chamber was calculated for spherically symmetric sources using the Green's function technique (Fujita *et al.* 1995; Sakuraba *et al.* 1995). Using the method presented in this paper, we can calculate the response of the system for any arbitrary source. For well-documented volcanic tremors such as those observed for the Aso volcano (Kaneshima *et al.* 1996), it may even be possible to perform a centroid moment tensor inversion to determine the location and mechanism of tremor sources, as is commonly done for earthquake source mechanism determinations in global seismology using normal modes of the Earth.

The application of the present method to the wave propagation problem in a layered half-space also seems straightforward and promising. This method requires computational effort to calculate the eigenfunctions. However, once we obtain and store them, it is straightforward to calculate the complete waveform (including body waves and surface waves) by summation of the eigenfunctions. This method may be more efficient than previously available methods, such as the reflectivity method, when we calculate the response of a stack of homogeneous layers, whose structure is well-determined, for various seismic sources.

The present method should be also useful when we consider the acoustic coupling between the atmosphere and the solid Earth, properly treating the outward radiating boundary in the modal approach. Such an approach has recently been taken by Lognonné, Clévédé & Kanamori (1998) based on the theory developed by Lognonné (1991), in which the biorthogonality relation is used to compute seismograms of a rotating and anelastic earth in terms of normal modes.

ACKNOWLEDGMENTS

We thank H. Kanamori for pointing out their recent work, and P. Lognonné for sending us a preprint of their paper.

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