Boundary Condition Identification based on 3D photoelasticity

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Objective

Final Goal

Governing mechanism of unstable crack growth
in a bulk body under compression

(For numerical simulation of earthquake fault behavior)

Current Target

Identification of state of stress in an
elastically deformed body using
3D photoelasticity

~~Identification of residual stress~~

Difficulty in 3D Photoelasticity

Nonlinear Inverse Problem:

observed photoelastic fringe patterns → state of stress

Nonlinearity

Load incremental approach

Measurement

Equations with relative phase

Robustness

Equilibrium conditions

Problem Setting

Governing Equation:

$$\frac{d\,\mathbf{A}}{d\,z} = \mathbf{G}\,\mathbf{A}$$

($\mathbf{A}$: Light vector traveling in z-direction)

$$\mathbf{A} = \begin{pmatrix} k_x e^{i\delta_x} \\ k_y e^{i\delta_y} \end{pmatrix}$$

k_x, k_y : amplitudes of light vector

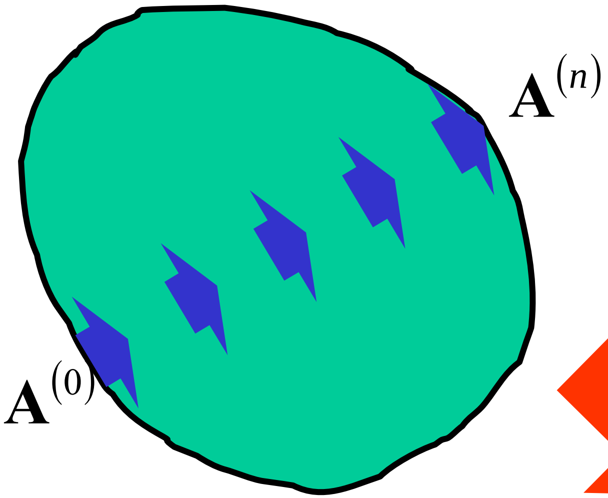
δ_x, δ_y : phases of light vector

$$\mathbf{G}(z) = iC_0 \begin{pmatrix} \sigma_{xx} - \sigma_{yy} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} - \sigma_{xx} \end{pmatrix}$$

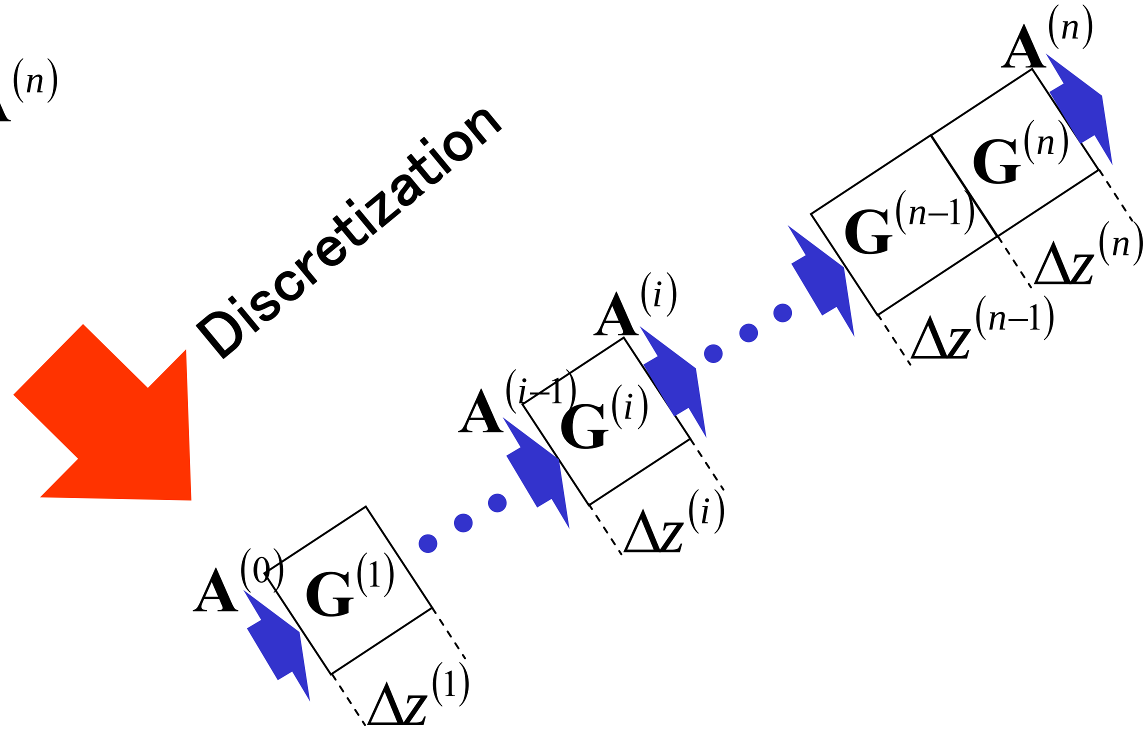
Problem: Identify the stress field $\boldsymbol{\sigma}$ using observed light vector $\mathbf{A}$

(where $\mathbf{A}$ contains the integrated effect of the stress field along the light ray)

Nonlinearity in 3D Photoelasticity



$$\frac{d\,\mathbf{A}}{d\,z} = \mathbf{G}\,\mathbf{A}$$



$$\mathbf{A}^{(n)} = \left(\mathbf{I} + \mathbf{G}^{(n)}\Delta z^{(n)}\right)\left(\mathbf{I} + \mathbf{G}^{(n-1)}\Delta z^{(n-1)}\right)\cdots\left(\mathbf{I} + \mathbf{G}^{(1)}\Delta z^{(1)}\right)\mathbf{A}^{(0)}$$

Output light is expressed as the non-commutable multiplicative form of the unknown matrices

Nonlinear Inverse Problem

What has been done to avoid nonlinearity?

$$\mathbf{A}^{(n)} = \left(\mathbf{I} + \mathbf{G}^{(n)} \Delta z^{(n)}\right)\left(\mathbf{I} + \mathbf{G}^{(n-1)} \Delta z^{(n-1)}\right) \cdots \left(\mathbf{I} + \mathbf{G}^{(1)} \Delta z^{(1)}\right) \mathbf{A}^{(0)}$$

Frozen Stress or Scattered Light

Mechanical or Optical slice

$$\mathbf{A}^{in} \Rightarrow \boxed{\mathbf{G}^{(i)}} \Rightarrow \mathbf{A}^{out}$$

Integrated Photoelasticity

$$\mathbf{A}^{(n)} = \left[\mathbf{I} + \sum_{i=1}^{n} \mathbf{G}^{(i)} \Delta z^{(i)}\right] \mathbf{A}^{(0)}$$

Limitation in size, material, state of stress

Linearization of the Governing Equation

$$\mathbf{A}^{(n)} = \left(\mathbf{I}+\mathbf{G}^{(n)}\Delta z^{(n)}\right)\left(\mathbf{I}+\mathbf{G}^{(n-1)}\Delta z^{(n-1)}\right)\cdots\left(\mathbf{I}+\mathbf{G}^{(1)}\Delta z^{(1)}\right)\mathbf{A}^{(0)}$$

Linearization w.r.t. infinitesimal stress increment

$$\begin{aligned}\dot{\mathbf{A}}^{(n)} = \Big[&\dot{\mathbf{G}}^{(n)}\Delta z^{(n)}\left(\mathbf{I}+\mathbf{G}^{(n-1)}\Delta z^{(n-1)}\right)\cdots\left(\mathbf{I}+\mathbf{G}^{(1)}\Delta z^{(1)}\right)\\ &+\left(\mathbf{I}+\mathbf{G}^{(n)}\Delta z^{(n)}\right)\dot{\mathbf{G}}^{(n-1)}\Delta z^{(n)}\cdots\left(\mathbf{I}+\mathbf{G}^{(1)}\Delta z^{(1)}\right)\\ &+\quad\vdots\qquad\qquad\vdots\\ &+\left(\mathbf{I}+\mathbf{G}^{(n)}\Delta z^{(n)}\right)\cdots\dot{\mathbf{G}}^{(2)}\Delta z^{(2)}\left(\mathbf{I}+\mathbf{G}^{(1)}\Delta z^{(1)}\right)\\ &+\left(\mathbf{I}+\mathbf{G}^{(n)}\Delta z^{(n)}\right)\cdots\left(\mathbf{I}+\mathbf{G}^{(2)}\Delta z^{(2)}\right)\dot{\mathbf{G}}^{(1)}\Delta z^{(1)}\Big]\dot{\mathbf{A}}^{(0)}\end{aligned}$$

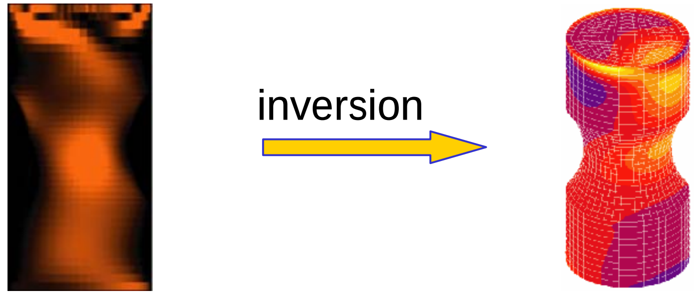
○ : Unknown

— : Known

$$\dot{\mathbf{A}}^{(n)} = \mathbf{X}^{(n)}\dot{\boldsymbol{\sigma}}^{(n)} + \mathbf{X}^{(n-1)}\dot{\boldsymbol{\sigma}}^{(n-1)} + \cdots + \mathbf{X}^{(2)}\dot{\boldsymbol{\sigma}}^{(2)} + \mathbf{X}^{(1)}\dot{\boldsymbol{\sigma}}^{(1)}$$

Linearization of the Governing Equation

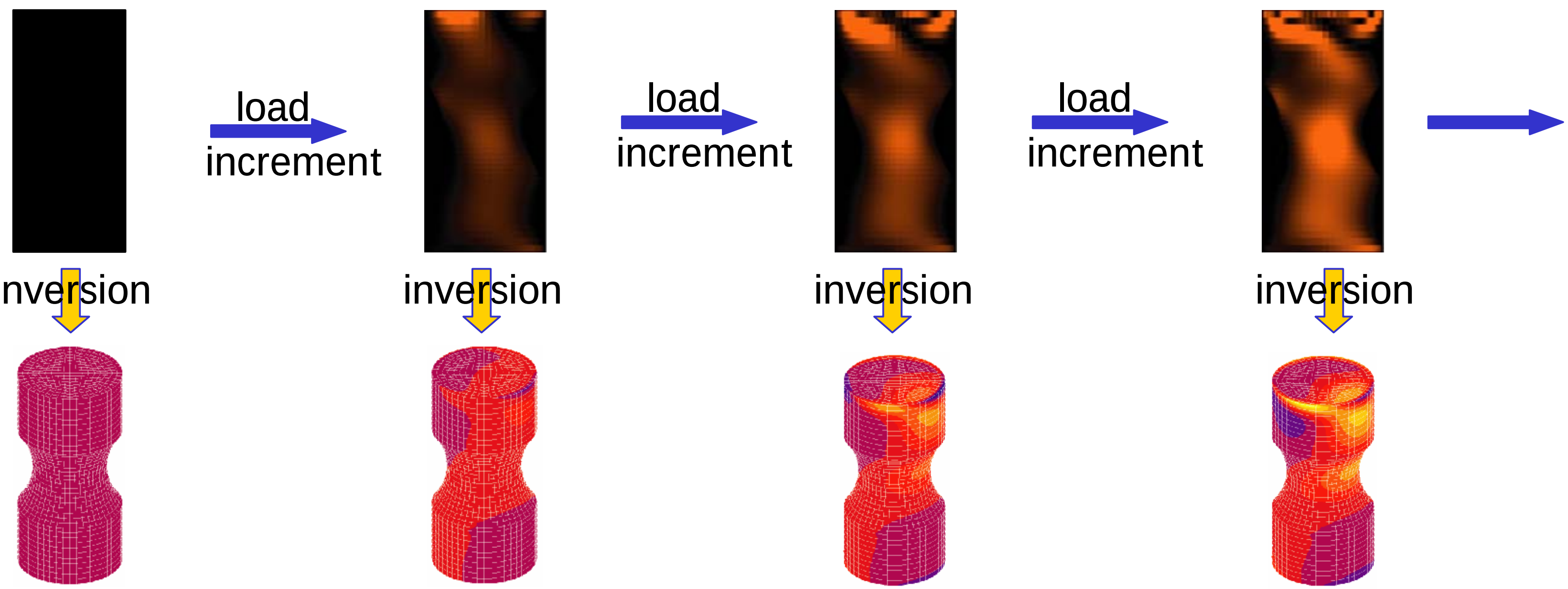
Conventional:



Impossible for general problems

Fringe patterns ⇨ State of stress

New:



Stress

Sequence of fringe patterns ⇨ Sequence of state of stress

Equation with Relative Phase Difference

$$\frac{d\,\mathbf{A}}{d\,z} = \mathbf{G}\,\mathbf{A} \qquad \mathbf{A} = \begin{pmatrix} k_x e^{i\delta_x} \\ k_y e^{i\delta_y} \end{pmatrix}$$

Needs: measurement of absolute phase shift

$$\left[k_x k_y \cos\delta\right]_{z_0}^{z_n} = C_0 \int_{z_0}^{z_n} \left(\sigma_{xx} - \sigma_{yy}\right) k_x k_y \sin\delta \, dz$$

$$\left[k_y^2 - k_x^2\right]_{z_0}^{z_n} = 4C_0 \int_{z_0}^{z_n} \sigma_{xy}\, k_x k_y \sin\delta \, dz$$

Complicated but free from measurement of absolute phase shift

Sensitivity to Measurement Noise

Error in measurements (%)	Global error (%)	Max. error in components (%)
0.1	0.002	21.3
0.5	0.058	106.7
1.0	0.247	233.3

$$\varepsilon_{\text{global}} = \frac{\sum \left(\sigma_{predict} - \sigma_{correct}\right)^2}{\sum \left(\sigma_{correct}\right)^2} \qquad \varepsilon_{\max} = \frac{Max.\left(\sigma_{predict} - \sigma_{correct}\right)}{\sigma_{correct}}$$

Identification of stress distribution near the crack tip

Need: robustness in local sense

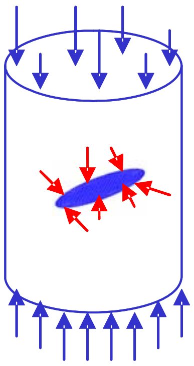
Introduction of Equilibrium Condition

Direct stress identification

- High sensitivity to measurement errors
- Huge DOF

Equilibrium condition
Linear elasticity

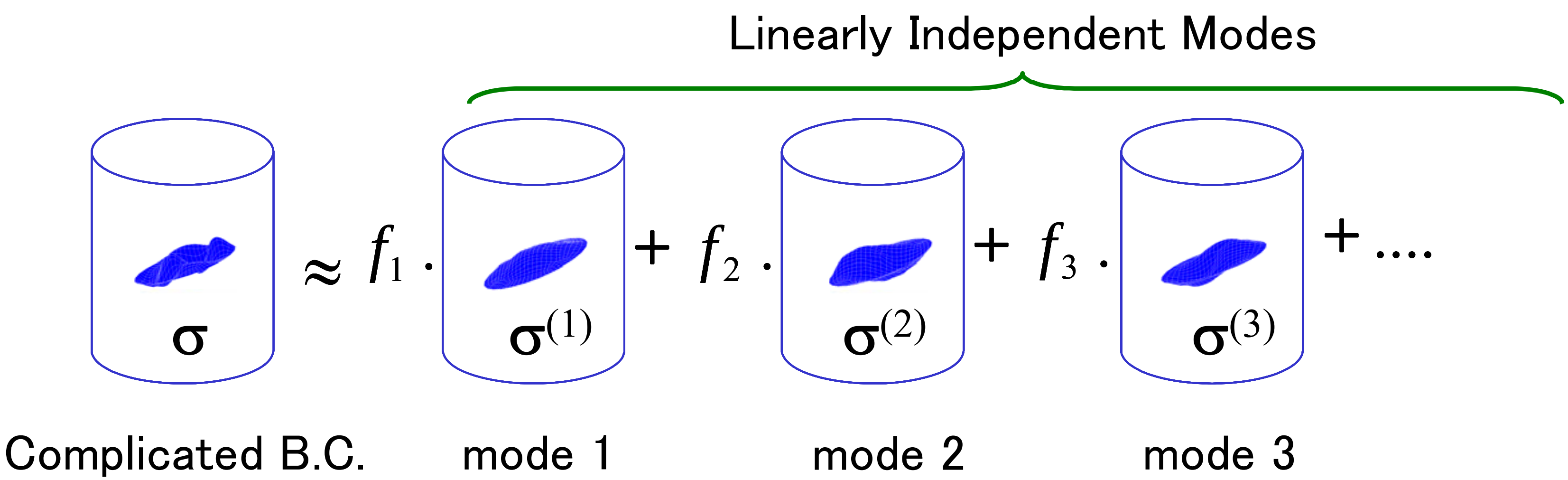
- Less sensitive to measurement errors
- Only unknown is boundary conditions



Linear elasticity → Trivial?

3D photoelasticity → Identification of Boundary Conditions

Identification of Boundary Conditions



Material : linearly elastic

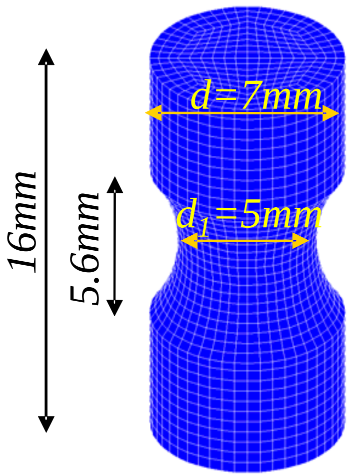
B.C. : linear combination of independent set of modes

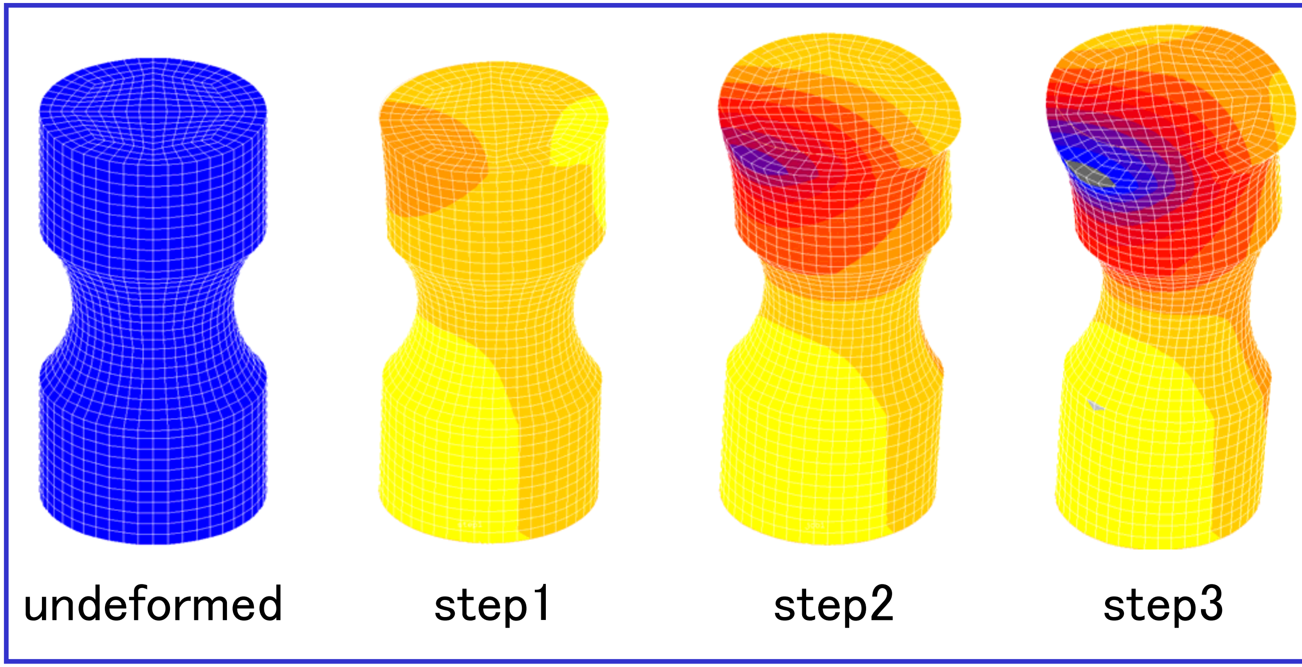
$\sigma(f_1, f_2, f_3 \ldots)$ —**nonlinear**→ Fringe patterns

↓ **Load incremental approach**

Δf_i —**linear**→ Increments of fringe patterns

Example Problem (problem setting)





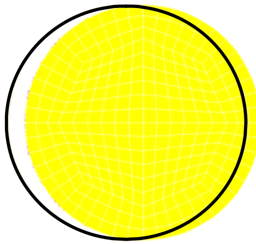
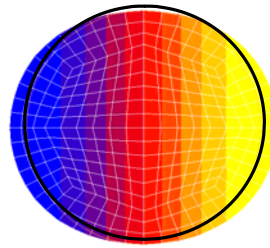
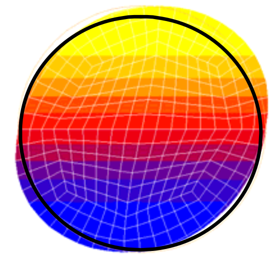
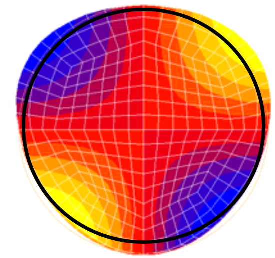
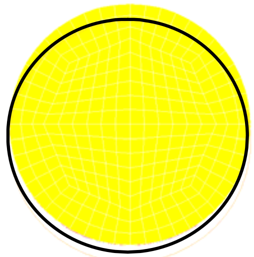
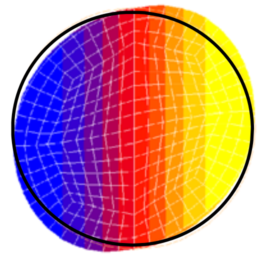
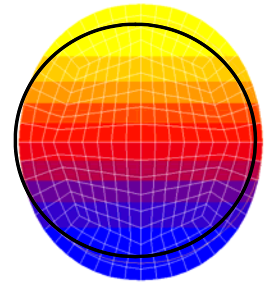
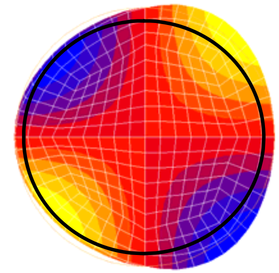
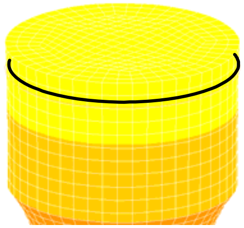
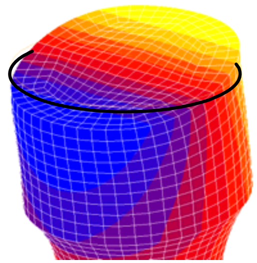
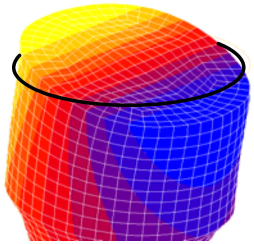
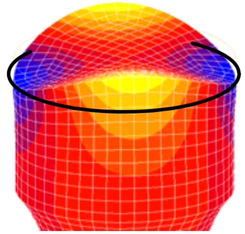
Material : Plexiglas

Boundary Conditions

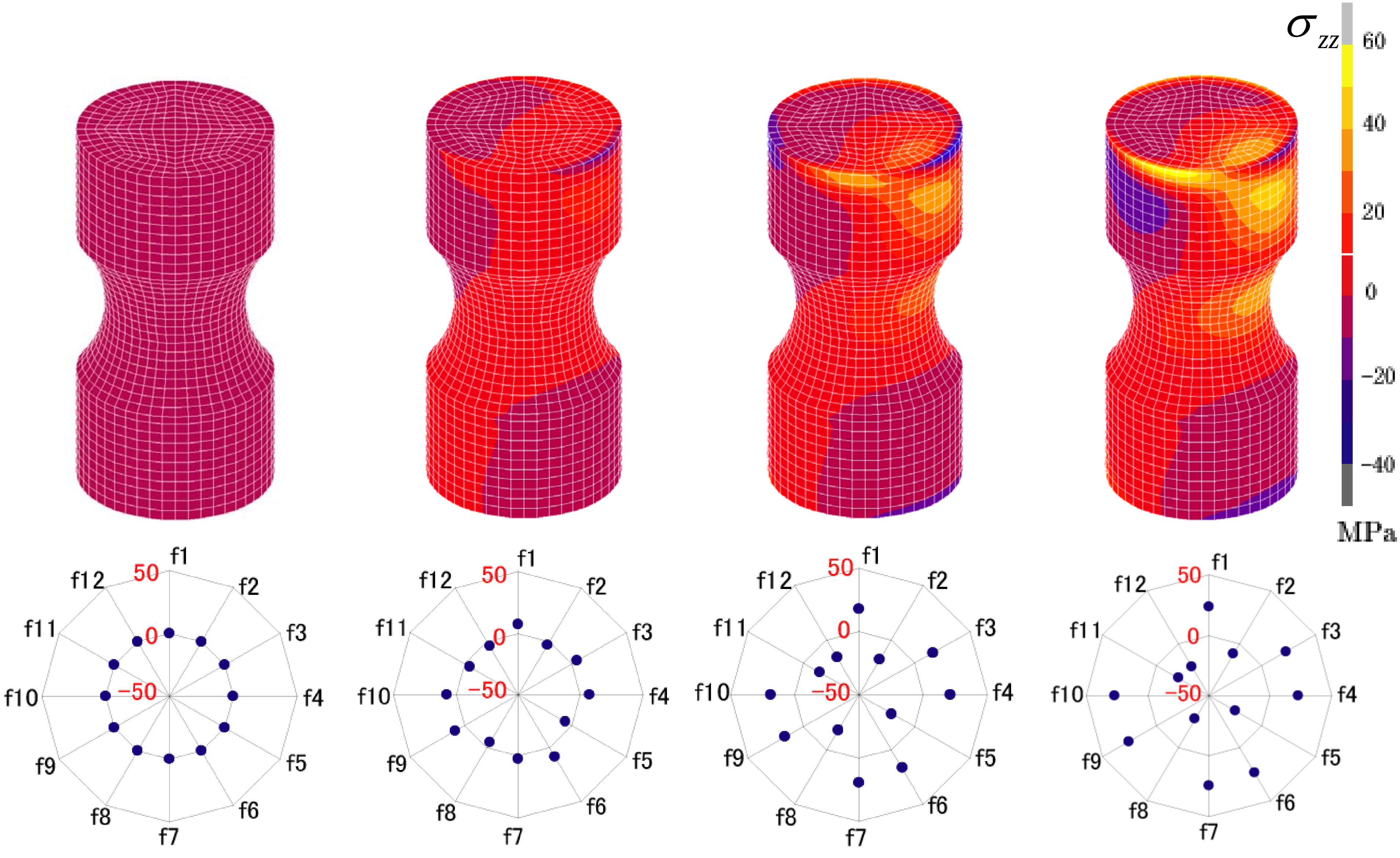
- Bottom face: fixed
- Top face: arbitrary combination of 4 modes in x, y and z directions (12unknowns)

Objective: identify boundary conditions based on output fringe patterns

Displacement Modes

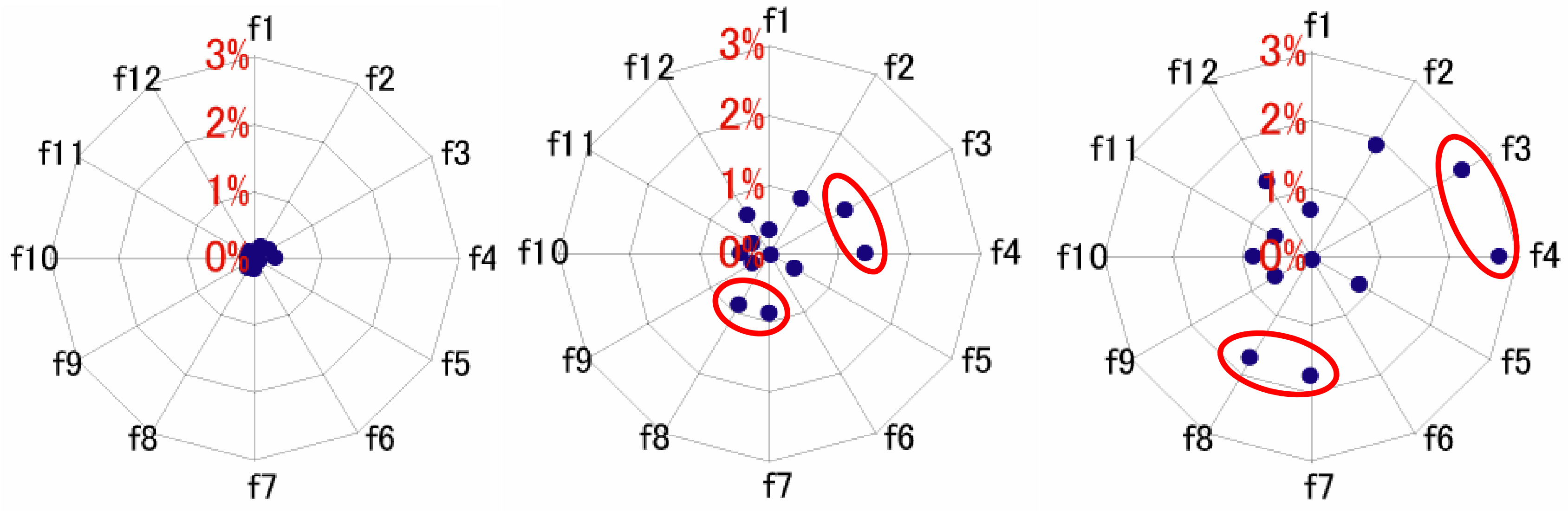
x	 mode 1, f1	 mode 2, f2	 mode 3, f3	 mode 4, f4
y	 mode 5, f5	 mode 6, f6	 mode 7, f7	 mode 8, f8
z	 mode 9, f9	 mode 10, f10	 mode 11, f11	 mode 12, f12

Example Problem (load steps)

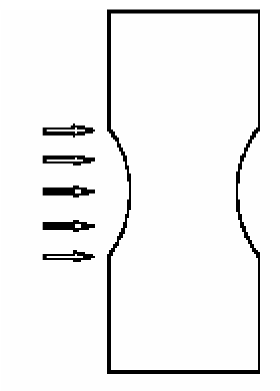


Step 0 (stress free) Step 1 Step 2 Step 3

Sensitivity to Measurement Noise
(mode amplitudes)



Error in measurements — 1 % — 5 % — 10 %



Higher shearing modes → Fast decay

Fast decaying modes are sensitive to measurement errors

Sensitivity to Measurement Noise

(stress components)

Error in measurements (%)	Global error (%)	Max. error in components (%)
1	0.0003	0.124
5	0.0059	1.003
10	0.0237	2.004

$$\varepsilon_{\text{global}} = \frac{\sum \left(\sigma_{predict} - \sigma_{correct}\right)^2}{\sum \left(\sigma_{correct}\right)^2} \qquad \varepsilon_{\max} = \frac{Max.\left(\sigma_{predict} - \sigma_{correct}\right)}{\sigma_{correct}}$$

Equilibrium condition

→ Robustness in prediction of local stress component

Summary

- Unstable crack growth in bulk body under compression

  Identification of state of stress in an elastically deformed body using 3D photoelasticity

- Incremental approach for 3D photoelasticity

  fringe pattern —nonlinear→ stress

  fringe pattern increment —linear→ stress increment

- Inverse analysis method applicable to experiment

  Governing equation with relative phase difference

- Direct stress identification
  - Highly sensitive to noise in measurements (stress increments are free to change)
  - Large number of DOF involved

- Boundary condition identification
  - Less sensitive to error in measurements (equilibrium condition)
  - Less number of DOF (equilibrium condition + linear elasticity)