INVERSE ANALYSIS METHODS OF IDENTIFYING CRUSTAL CHARACTERISTICS USING GPS ARRYA DATA

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Contents

1. Stress inversion method: find equilibrating stress using measured strain and partial information on stress-strain relation
2. Elasticity inversion method: find local elasticity using densely measured displacement
NEED FOR LOCAL STRESS PREDICTION

- Material Test of Next Generation
- Earthquake Prediction

![Diagram](image.png)

- Local $\varepsilon-\sigma$ relation
- Regional stress
- GPS data
- Strain
IS STRESS INVERSION POSSIBLE?

- 3D State: Most Difficult
- 2D State: Possible?

3 UNKNOWNS

\[
\begin{align*}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{12}
\end{align*}
\]

2 EQUATIONS

- Equilibrium in \(x_1\)-direction
- Equilibrium in \(x_2\)-direction

1 CONDITION

STRAIN
STRESS INVERSION

Airy

\begin{align*}
\sigma_{11} &= a_{,22} \\
\sigma_{22} &= a_{,11} \\
\sigma_{12} &= -a_{,12}
\end{align*}

self-equilibrating stress

partial information: \( \sigma_{11} + \sigma_{22} = \mathbf{f}(\varepsilon_{ij}) \)

\[ \mathbf{f}(\varepsilon_{ij}) = \kappa(\varepsilon_{11} + \varepsilon_{22}) \]

\[ \sigma_{11} + \sigma_{22} = \kappa(\varepsilon_{11} + \varepsilon_{22}) \]

2D bulk modulus?

B.V.P.

Poisson equation

G.E. \( a_{,11} + a_{,22} = \kappa(\varepsilon_{11} + \varepsilon_{22}) \)

B.C. \( n_1 a_{,1} + n_2 a_{,2} = -n_1 r_2 + n_2 r_1 \)

measured strain

resultant forces
EXTENSION TO OTHER DEFORMATION STATE

◆ Dynamic State

\[
\begin{aligned}
\sigma_{11,1} + \sigma_{12,2} &= \rho \ddot{u}_1 \\
\sigma_{12,1} + \sigma_{22,2} &= \rho \ddot{u}_2 \\
\sigma_{11} + \sigma_{22} &= f(\varepsilon)
\end{aligned}
\]

◆ Finite Deformation State:

\[
\begin{aligned}
\sum_{j,k} \left( \frac{\partial X_k}{\partial x_j} \right) \sigma_{1j,k} &= 0 \\
\sum_{j,k} \left( \frac{\partial X_k}{\partial x_j} \right) \sigma_{2j,k} &= 0 \\
\sigma_{11} + \sigma_{22} &= f(\varepsilon)
\end{aligned}
\]

\(\sigma_{ij} = \sigma_{ij}(X), \quad x_i = x_i(X)\)
NUMERICAL SIMULATION

- Conditions
  - elasto-plastic material with unknown yield function
  - prediction of stress and stress-strain relation

FEM computation with 20x48 elements

displacement field is used as input data
RESULTS OF INVERSION

distribution of $\sigma_{12}$

exact

sample surface

predicted

principle stress-
plastic strain rate

$\sigma_1 - \sigma_3$

$\sigma_2 - \sigma_3$

exact

yield locus

predicted

good agreement
MODEL EXPERIMENT

Torsional Shearing

Riedel shears

video recorder

CCD camera

servomotor

Experiment Apparatus

max. shear strain

1.8–2
1.6–1.8
1.4–1.6
1.2–1.4
1–1.2
0.8–1
0.6–0.8
0.4–0.6
0.2–0.4
0–0.2

examples of image

7cm

8cm
OVERALL STRESS-STRAIN RELATION

will be different from local relations?
LOCAL STRESS-STRAIN RELATIONS

A: far from crack
D: near crack

no common relations?
RESULTS OF INVERSION

common elasto-plastic relations?
APPLICATION TO GPS ARRYA DATA

- verification of numerical analysis method
  - check numerical stability of solving boundary value problem
  - check dependency of parameters

- application of stress inversion method
  - geophysical interpretation of analysis results
  - critical examination of assumption of plane state

- development of crust deformation monitor
  - automatic processing of GPS array data
CONVERGENCE

σ (hydrostatic stress)

Δ: resolution of strain distribution (degree)
EFFECT OF REFERENCE

\( \tau \) (max. shear stress)

a) \( v=0.1 (\tau) \)

b) \( v=0.2 (\tau) \)

c) \( v=0.3 (\tau) \)

d) \( v=0.4 (\tau) \)
COMPARISON OF STRESS WITH STRAIN

σ (hydrostatic stress)

τ (max. shear stress)

ε (vol. strain)

γ (max. shear strain)
REGIONAL CONSTITUTIVE RELATIONS

regional stiffness
($\tau, \gamma$: max. shear stress and strain)

regional anisotropy
($\phi, \gamma$: principle stress and strain)

regional heterogeneity and anisotropy
CHANGE IN INVARIANT

1st invariant

98/12/04

2nd invariant

98/12/04
CHANGE IN REGIONAL STATE

1st invariant

2nd invariant

A

B

C

ε

σ

γ

τ

ε

σ

γ

τ
REGIONAL STRESS AND STRAIN

1st invariant

\[ \varepsilon - \gamma \]

2nd invariant

\[ \gamma - \tau \]
GPS DATA DURING 1998-1999

◆ GPS Data
  – no spatial filtering to get rid of measurement noise
  – linear interpolation between two GPS station

◆ More Sophisticated Treatment of BVP
  – FEM with triangle element
  – weak form
  – regionally averaged field quantities
BVP in Rate Form and Weak Form

G.E. \( \dot{\alpha}_{11} + \dot{\alpha}_{22} = \kappa (\dot{\epsilon}_{11} + \dot{\epsilon}_{22}) \)

B.C. \( n_1 \dot{\alpha}_{1} + n_2 \dot{\alpha}_{2} = -n_1 \dot{r}_2 + n_2 \dot{r}_1 \)

\[
\int \varphi_{11} \dot{\alpha}_{1} + \varphi_{22} \dot{\alpha}_{2} - \kappa (\varphi_{11} \dot{u}_1 + \varphi_{22} \dot{u}_2) dS = 0
\]

Computation of Average Quantities

- Average stress rate computed by using 1st-order derivative

\[
\langle \dot{\epsilon}_{11} \rangle = \frac{1}{\Omega} \int_{\partial\Omega} n_1 \dot{u}_1 dL
\]

\[
\langle \dot{\sigma}_{11} \rangle = \frac{1}{\Omega} \int_{\partial\Omega} n_2 \dot{\alpha}_{2} dL
\]
REGIONAL STRAIN RATE

volumetric

max. shear
COMPARISON WITH SEISMIC EVENTS?

volumetric

>20[km], >M3
find $\kappa(x)$ s.t.

$$\tau(x; \kappa) = \kappa(x)\dot{\gamma}(x)$$

$\tau, \dot{\gamma}$ maximum shear
$\kappa$ regional stiffness

$\kappa$ is originally used to relate $\sigma$ & $\varepsilon$ through $\sigma = \kappa \varepsilon$.

not too far from known geological structure
DRAWBACKS OF STRESS INVERSION

◆ Need to Know One Constitutive Relation
  – bulk stress and bulk strain
  – isotropy assumption

◆ Need to Know Boundary Traction/Resultant Force
  – assumption of uniform stress
  – fast decrease of non-uniform boundary traction

◆ Difficulty in Understanding Plane-Stress-State Model

another analysis method needed?
DRAWBACKS OF ELASTICITY INVERSION

- Sensitive to Displacement Error
  - need to make fine discretization of target body
  - need to have some strong modes of deformation

Why is it so?
- no mistakes in mathematics
- poor understanding of physics
May not be good to pose an inverse problem from data to characteristics, because a path from characteristics to data has physical process and measurement.
estimation: from data to response, i.e., estimate function for displacement from data which are measured discretely

inversion: from response to characteristics, i.e., find most suitable characteristics for physical process even though source is not known
nodes at which displacement is measured

inner node

boundary node

block S

displacement at all nodes given

no need to consider interaction of S with outside region
IDENTIFICATION OF DISPLACEMENT MODE

- apply several BC’s, and measure displacement at nodes of a hexagonal block.
  1. identify displacement modes (a characteristic set of nodal displacement)
  2. identify local elasticity
IDENTIFICATION OF DISPLACEMENT MODE

Displacement mode and elastic parameters diagram.
ELASTICITY INVERSION METHOD

1. Use displacement data to determine Taylor series expansion coefficients of displacement.

2. Use equilibrium equation to estimate elastic parameters by expanding stress in Taylor series.

BASIC PROCEDURES OF INVERSION

- Local field variables allow Taylor series expansion.
- Uniform Poisson's ratio is assumed.

Diagram:
- Body \( B \)
- Block \( \Omega \)
- Nodes at which displacement is measured.
DETERMINATION OF DISPLACEMENT COEFFICIENT

1. Taylor Expansion: \( \{a_{ip}\} \)

\[
u_i(x) = \sum_{p=1}^{P} a_{ip} f_p(x) \quad \{a_{ip}\} = \{u_i, u_{i,1}, u_{i,2}, \frac{1}{2} u_{i,2}^2, u_{i,11}, \frac{1}{2} u_{i,12}, u_{i,22}, \cdots\}
\]
\[
\{f_p\} = \{1, x_1, x_2, x_1^2, x_1 x_2, x_2^2, \cdots\}
\]

2. Displacement Data: \( \{\overline{u}_i^n\} \)

\[
\overline{u}_i^n = \sum_{p=1}^{P} f_{pn} a_{ip}
\]

3. Solution of Matrix Equation \( f_{pn} = f_p(x^n) \)

\[
a_{ip} = \sum_{\alpha=1}^{A} \frac{1}{\lambda^\alpha} \left( \sum_{n} u_i^n \varphi_n^\alpha \right) \psi_p^\alpha + \sum_{\beta=1}^{P-A} b_\beta \psi_p^\beta \quad \{\lambda^\alpha, \varphi_n^\alpha, \psi_p^\alpha\} : \text{SVD of } f_{pn}
\]

fully determined \quad undetermined
1. Elasticity Tensor Expressed in Terms of Poisson Ratio

\[ c_{ijkl} = c^0_{ijkl} + \nu c^1_{ijkl} \]

2. Equation of Equilibrium and Its Taylor Expansion

\[ b_i(x) = (c_{ijkl} u_{k, l}(x))_{, j} = \sum_p b_{ip} f_p(x) = 0 \]

3. Coefficient of Expansion: \( b_{ip} = 0 \) for 0\textsuperscript{th} Order (\( p=1 \))

\[
\begin{bmatrix}
  b_{11} \\
  b_{21}
\end{bmatrix} = \begin{bmatrix}
  2 & 0 & 0 & \frac{1}{2} & 1 & 0 \\
  0 & 1 & \frac{1}{2} & 0 & 0 & 2
\end{bmatrix} + \nu \begin{bmatrix}
  0 & 0 & 0 & \frac{1}{2} & 1 & 0 \\
  0 & -1 & \frac{1}{2} & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
  a_{14} \\
  a_{24} \\
  \vdots \\
  a_{26}
\end{bmatrix}
\]

linear equation of \( \nu \) is derived
NUMERICAL SIMULATION (1)

**Graphs:**
- Error vs. $\nu$ for 2nd and 3rd order expansions.
  - $\nu = 0.25$: 2nd order error is higher than 3rd order error.
  - $\nu = 0.35$: Error is negligible for both orders.

**Tables:**
- Measurement: 200
- Expansion of displacement: 3rd order
- Expansion of equilibrium: 0th or 1st order

**Charts:**
- Frequency vs. $\nu$ for $\nu = 0.25$ and $\nu = 0.35$.
1. Measurement Error: \( \{e_i^n\} \)

\[
\overline{u}_i^n = \sum_{p=1}^P f_{pn} a_{ip} + e_i^n
\]

2. Find \( \nu \) such that

\[
\text{minimize } |e|^2 = \sum (e_i^n)^2
\]

subjected to \( b_{ip}(\nu) = 0 \)

- measurement 200
- expansion of displacement 3\text{rd} order
- expansion of equilibrium 0\text{th} or 1\text{st} order
APPLICATION OF LOCAL GPS DATA

2000 Western Tottori Earthquake (M_{JMA}=7.3) examine change in deformation and elasticity before and after this earthquake

- 53 GPS observation points
- 82 triangular elements
- GPS data obtained from 1997 to 2002
- annual and biannual sinusoidal variations excluded
STRAIN RATE

**pre-WTE**

**dilatational**

**post-WTE**

**Maximum shear strain rate**

97.05.24 -- 00.08.06

00.11.04 -- 02.07.27

**Maximum shear strain rate**

97.05.24 -- 00.08.06

00.11.04 -- 02.07.27

**Dilatational strain rate**

97.05.24 -- 00.08.06

00.11.04 -- 02.07.27

**shear**
STRESS RATE

pre-WTE

dilatational

Dilatational stress increment
97.05.24 -- 00.08.06

Maximum shear stress increment
97.05.24 -- 00.08.06

post-WTE

Dilatational stress increment
00.11.04 -- 02.07.27

Maximum shear stress increment
00.11.04 -- 02.07.27

shear
Principal axes of strain rate
97.05.24 -- 00.08.06
(extensional)
(contractional)
0.5 [Micro Strain/Year]

Principal axes of strain rate
00.11.04 -- 02.07.27
(extensional)
(contractional)
0.5 [Micro Strain/Year]

Principal axes of stress rate
97.05.24 -- 00.08.06
(extensional)
(contractional)
100 [KPa/Year]

Principal axes of stress rate
00.11.04 -- 02.07.27
(extensional)
(contractional)
100 [KPa/Year]
Poisson’s ratio

Post-Tottori Earthquake

Poisson’s ratio

Post-Tottori Earthquake

Change of Poisson’s ratio

Post-Tottori Earthquake

Poisson’s ratio

Pre-Tottori Earthquake

Poisson’s ratio

Pre-Tottori Earthquake

Change of Poisson’s ratio

Pre-Tottori Earthquake

\( \nu \) has been reduced near the source faults. The comparison with other analyses are being made.
CONCLUDING REMARKS

◆ Two inverse analysis methods
  – stress inversion
    find Airy’s stress function by solving Poisson’s equation
  – elasticity inversion
    find elastic parameters by estimating displacement expansion coefficients

◆ Development of new inversion is needed for geophysics where experiments cannot be made.

◆ Application
  – small material samples used for bio-mechanics
  – geomaterials
  – new image analysis with higher spatial resolution