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# Plane-wave and Flat Earth Approximations in Natural-source Electromagnetic Induction Studies

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**Abstract**

When applying electromagnetic sounding methods, such as the magnetotelluric method, studies are usually carried out by: (1) treating the inducing field as spatially uniform and (2) treating the Earth as a semi-infinite conductor with a plane surface. These assumptions are both approximations of electromagnetic induction caused by the incidence of a laterally non-uniform inducing field into the conducting spherical Earth. Although the basic theoretical concept was established many decades ago, the physical conditions for these two approximations are not fully and systematically understood, and some confusion appears in the literature. Therefore, the basic formulation of electromagnetic induction in both spherical and Cartesian coordinate systems is re-examined and the conditions for systematically deriving the two approximations are clarified. The results reveal that the solutions for the two coordinate systems are consistent with each other at an appropriate limit and that the two approximations result in neither indefinite nor non-unique problems, as suggested by some previous studies, if appropriate approximation conditions are applied.

**Keywords**: electromagnetic induction, magnetotelluric method, plane-wave approximation, flat Earth approximation, electrical conductivity

## 1. Introduction

The magnetotelluric (MT) method is a geophysical exploration tool that is widely used for various purposes, such as mineral explorations, crustal studies, and deep mantle studies (e.g., Chave and Jones, 2012). The basic theory behind the MT method (Cagniard, 1953) relies on electromagnetic (EM) induction in the Earth, which is assumed to be a half-space with a flat surface consisting of electrically conducting materials with an arbitrary distribution, due to the incidence of spatially uniform inducing field variations from outside (originating in the ionosphere and/or magnetosphere). Here, we consider two approximations to be important to the theoretical framework. One approximation allows us to treat the Earth's actually spherical surface as if it were flat, which we hereinafter refer to as the flat Earth approximation, and the other approximation allows us to treat the external inducing field as spatially uniform, which we hereinafter refer to as the plane-wave approximation. Both are obviously approximations, because, in reality, neither is the Earth's surface flat nor are external EM fluctuations spatially uniform. These two approximations in the EM theory are the main topic of the present paper. Although every EM induction method, including the MT method, relies on other approximations, such as, for example, neglecting the displacement current in Maxwell equations, they are beyond the scope of the present paper. In addition, deviations of the Earth's geometry from a perfect sphere are ignored.

Before the first publication by Cagniard (1953), Price (1950) presented a more general theory of EM induction using a flat Earth approximation without a plane-wave approximation. These two pioneering studies were followed by a long-lasting controversy as to whether the spatial non-uniformity of the inducing field should be taken into account and/or how significant its effects would be (Wait, 1954; Price, 1962; Srivastava, 1965; Towle, 1974). After this controversy, the method proposed by Cagniard (1953) based on a plane-wave approximation was successfully developed, but much less attention has been focused on the problem of the source effect, insofar as EM variations observed at relatively short periods (high frequencies) and/or at mid-low latitudes are concerned. Conversely, it is generally recognized that careful treatment may be required when MT data are acquired at high latitudes (Viljanen *et al.*, 1999; Pulkkinen

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*et al.*, 2003) or in the period band where Solar quiet daily geomagnetic (Sq) variations are dominant (Schmucker, 1999; Shimizu *et al.*, 2011; Koch and Kuvshinov, 2015). Most recently, Murphy and Egbert (2018) reported that the effects of a finite source scale may be significant, even within a period range of 10 to 100 s for geomagnetic pulsations Pc3 to Pc4 at mid-low latitudes. This suggests that the theory of Price (1950) may still be useful for MT studies on particular occasions when external EM field fluctuations are localized.

For periods longer than a few days, the EM induction problem is treated in a spherical coordinate system (e.g., Banks, 1969), because the Earth's sphericity cannot be ignored in such a situation. The approach is referred to as global induction when the electrical structure of the entire Earth is considered, or semi-global induction when the heterogeneous structure of only a limited region is considered (Utada *et al.*, 2003). Correspondingly, an approach using a flat Earth approximation with solutions of Maxwell equations in a Cartesian coordinate system is referred to as regional (or local) induction, which is applied for periods less than approximately $10^4$ s. In each approach, EM response functions, which are defined as the ratio of different components of EM field variations or the ratio of different coefficients of spherical harmonic expansion, are calculated from observed time-series data to estimate the electrical conductivity distribution of the Earth. If a flat Earth approximation is allowed, the equivalent response functions defined in global and regional induction approaches are expected to be consistent at an appropriate limit, because they are not distinguishable (Weaver, 1994). However, a few studies argue that the values of some response functions are indeterminate (Price, 1950; 1962; Honkura and Rikitake, 1985) or non-unique (Weaver, 1994), as discussed later.

The present paper briefly reviews these two basic approximations in the EM induction problem to aid a systematic understanding of the theory, including some knowledge that was a matter of serious controversy but has almost been forgotten. Because the main focus of the present paper is the approximate treatment of the geometry of the external inducing field (spatially uniform or non-uniform) and of the Earth's surface (flat or spherical), we assume the simplest model of the Earth's structure, in which the Earth is treated as a uniform conducting sphere with the global induction approach and as a uniform half-space with the regional induction approach. In other words, the validity of both approximations relies mostly on the features of the primary field that reflects the average structure, but is much less affected by the features of the secondary field that reflects the lateral heterogeneity of the Earth's conductivity (Lezaeta *et al.*, 2007). Schmucker (1985) conducted a more comprehensive and detailed investigation than the present paper on the behaviors of EM response functions derived both in a Cartesian coordinate system and in a spherical coordinate system. However, some of the basic derivation processes or conditions of the approximation are not clearly shown. Thus, another goal of the present paper is to provide a basic formulation from the very beginning to better understand the EM induction problem.

## 2. Basic theory of EM induction in the Earth with uniform conductivity

A number of studies in the literature describe the basic theory of EM induction, but few describe both global and regional induction approaches systematically. Here, for the convenience of a later discussion, let us summarize the basic theory of EM induction in the simplest situation (uniform conductivity) for both approaches. We choose spherical (Fig. 1a) and Cartesian (Fig. 1b) coordinate systems, which are common in electromagnetism (Langel, 1987). For the theoretical derivation, we consider Maxwell equations in a frequency domain with angular frequency denoted by $\omega$ and ignore the displacement current,

$$\nabla \times \mathbf{E}(\mathbf{r}, \omega) = -i\omega\, \mathbf{B}(\mathbf{r}, \omega), \tag{1}$$

$$\nabla \times \mathbf{H}(\mathbf{r}, \omega) = \sigma \mathbf{E}(\mathbf{r}, \omega) + \mathbf{j}_{ext}(\mathbf{r}, \omega), \tag{2}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}, \omega) = 0, \tag{3}$$

where $\mathbf{j}_{ext}$ is the source current, which produces the inducing field. We denote electrical conductivity in the Earth as $\sigma$, which is assumed to be constant inside the Earth. Magnetic field, magnetic induction, and electric field are denoted as $\mathbf{H}(\mathbf{r}, \omega)$, $\mathbf{B}(\mathbf{r}, \omega)$, and $\mathbf{E}(\mathbf{r}, \omega)$, respectively. Magnetic permeability $\mu$ is assumed to be constant, so we have $\mathbf{B}(\mathbf{r}, \omega) = \mu \mathbf{H}(\mathbf{r}, \omega)$. The position vector is given as $\mathbf{r} = (r, \theta, \varphi)^t$ in a spherical system and $\mathbf{r} = (x, y, z)^t$ in a Cartesian system, where the superscript $t$ denotes the transpose.

In EM induction, the source is assumed to be outside the Earth, and the space between the source and

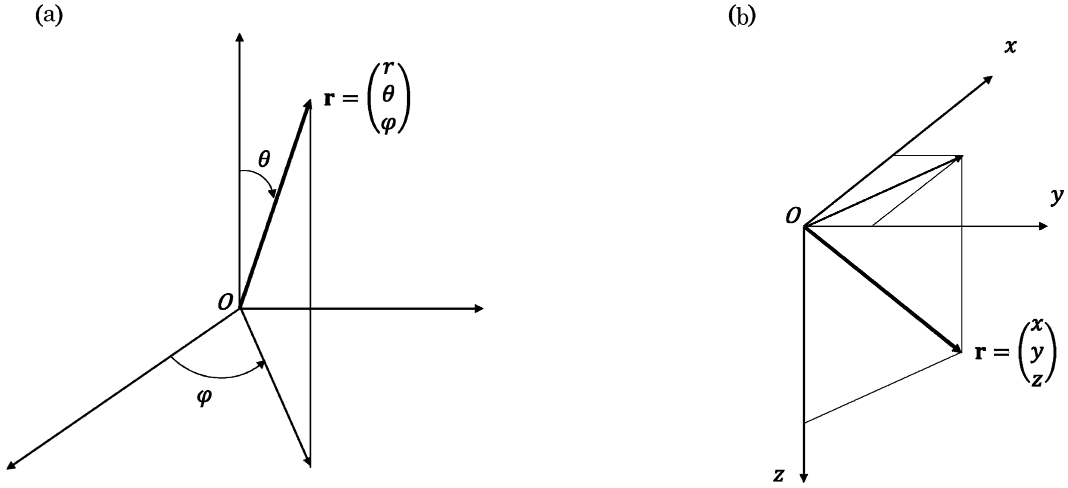


Fig. 1. (a) Spherical coordinate system and (b) Cartesian coordinate system used in the present study. The center of the Earth is chosen as the origin ($O$) in (a), while a certain point on the Earth's surface is chosen in (b).

the Earth's surface is assumed to be an insulator ($\sigma=0$). Therefore, from (2), in this insulating space, we have

$$\nabla\times\mathbf{H}(\mathbf{r},\omega)=\frac{1}{\mu}\nabla\times\mathbf{B}(\mathbf{r},\omega)=0, \tag{4}$$

which allows us to define the magnetic scalar potential $\psi(\mathbf{r},\omega)$,

$$\mathbf{B}(\mathbf{r},\omega)=-\nabla\psi(\mathbf{r},\omega). \tag{5}$$

From (3) and (5), we have the Laplace equation for the scalar potential outside the Earth,

$$\nabla^2\psi(\mathbf{r},\omega)=0. \tag{6}$$

Because $\mathbf{j}_{ext}=0$ in (2) inside the Earth, we can represent

$$\mathbf{B}(\mathbf{r},\omega)=\nabla\times\mathbf{\Pi}(\mathbf{r},\omega), \tag{7}$$

to satisfy (3), where $\mathbf{\Pi}$ is the vector potential. Using the Coulomb gauge,

$$\nabla\cdot\mathbf{\Pi}(\mathbf{r},\omega)=0, \tag{8}$$

the electric field is given by,

$$\mathbf{E}(\mathbf{r},\omega)=-i\omega\,\mathbf{\Pi}(\mathbf{r},\omega). \tag{9}$$

The vector potential satisfies the following basic equation:

$$(\nabla^2-i\omega\sigma\mu)\mathbf{\Pi}(\mathbf{r},\omega)=0. \tag{10}$$

Because of condition (8), the vector potential can be expressed by two independent modes, poloidal and toroidal (magnetic) modes (e.g., Backus, 1986), which can take the form

$$\mathbf{\Pi}(\mathbf{r},\omega)=\nabla\times(\mathbf{r}\chi(\mathbf{r},\omega))+\nabla\times\nabla\times(\mathbf{r}\chi'(\mathbf{r},\omega)), \tag{11}$$

in the case of a global approach, and

$$\mathbf{\Pi}(\mathbf{r},\omega)=\nabla\times(\hat{z}\chi(\mathbf{r},\omega))+\nabla\times\nabla\times(\hat{z}\chi'(\mathbf{r},\omega)), \tag{12}$$

in the case of a regional approach, where $\hat{z}$ is a unit vector in the vertical direction. The scalar functions $\chi$ and $\chi'$ satisfy (e.g., Banks, 1969)

$$(\nabla^2-i\omega\sigma\mu)\chi(\mathbf{r},\omega)=0, \tag{13}$$

$$(\nabla^2-i\omega\sigma\mu)\chi'(\mathbf{r},\omega)=0. \tag{13$'$}$$

Because of the formal identity of (13) and (13)′, these equations should have a common general solution with different coefficients, which are determined by applying appropriate boundary conditions. We hereinafter denote all variables for the toroidal mode with single prime symbols. In considering EM induction problems due to an external inducing field, the source (inducing) field is given as a coefficient in the expression of the scalar potential.

### 2.1 Solution for global/semi-global induction in a spherical coordinate system

A general solution for (6) in a spherical coordinate system can be obtained by separating variables in the form of spherical harmonic expansion (e.g., Langel, 1987),

$$\psi(\mathbf{r},\omega)=a\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\left[\epsilon_n^m(\omega)\left(\frac{r}{a}\right)^n+\iota_n^m(\omega)\left(\frac{a}{r}\right)^{n+1}\right]Y_n^m(\theta,\varphi), \tag{14}$$

where $Y_n^m$ is a spherical harmonic function of degree $n$ and order $m$, and $a$ is the Earth's radius. In addition, $\epsilon_n^m(\omega)$ and $\iota_n^m(\omega)$ are complex-valued expansion coefficients of the external (inducing) and internal (induced) parts of the scalar potential, respectively. Furthermore, we assume the Earth to be a perfect sphere, because the argument of the deviation of its actual geometry from a perfect sphere is not essential for the present purpose. Using (5) and (14), the three components of the magnetic induction **B** outside the conducting sphere ($r>a$) can be derived as follows:

$$B_r(\mathbf{r},\omega) = -\sum_{n,m}\left[n\epsilon_n^m(\omega)\left(\frac{r}{a}\right)^{n-1} - (n+1)\iota_n^m(\omega)\left(\frac{a}{r}\right)^{n+2}\right]Y_n^m(\theta,\varphi), \tag{15}$$

$$B_\theta(\mathbf{r},\omega) = -\sum_{n,m}\left[\epsilon_n^m(\omega)\left(\frac{r}{a}\right)^{n-1} + \iota_n^m(\omega)\left(\frac{a}{r}\right)^{n+2}\right]\frac{\partial}{\partial\theta}Y_n^m(\theta,\varphi), \tag{16}$$

$$B_\varphi(\mathbf{r},\omega) = -\sum_{n,m}\left[\epsilon_n^m(\omega)\left(\frac{r}{a}\right)^{n-1} + \iota_n^m(\omega)\left(\frac{a}{r}\right)^{n+2}\right]\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}Y_n^m(\theta,\varphi). \tag{17}$$

Note that the double summation in (14) is simplified in (15) through (17), and we use the same notation hereinafter.

A general solution for (13) and (13)′ can be derived with a variable separation as

$$\chi(\mathbf{r},\omega) = \sum_{n,m}R_n^m(r,\omega)\,Y_n^m(\theta,\varphi) \tag{18}$$

$$\chi'(\mathbf{r},\omega) = \sum_{n,m}R_n^{m\prime}(r,\omega)\,Y_n^m(\theta,\varphi), \tag{18$'$}$$

and the radial functions $R_n$ and $R_n'$ satisfy the same differential equation

$$\frac{d^2R_n^m}{dr^2} + \frac{2}{r}\frac{dR_n^m}{dr} + \left\{k^2 - \frac{n(n+1)}{r^2}\right\}R_n^m = 0, \tag{19}$$

and

$$\frac{d^2R_n^{m\prime}}{dr^2} + \frac{2}{r}\frac{dR_n^{m\prime}}{dr} + \left\{k^2 - \frac{n(n+1)}{r^2}\right\}R_n^{m\prime} = 0. \tag{19$'$}$$

The general solution for (19) and (19)′ can be written as (Banks, 1969),

$$R_n^m(r,\omega) = \alpha_n^m(\omega)j_n(kr) + \beta_n^m(\omega)y_n(kr), \tag{20}$$

and

$$R_n^{m\prime}(r,\omega) = \alpha_n^{m\prime}(\omega)j_n(kr) + \beta_n^{m\prime}(\omega)y_n(kr), \tag{20$'$}$$

where $\alpha_n^m(\omega)$, $\beta_n^m(\omega)$, $\alpha_n^{m\prime}(\omega)$, and $\beta_n^{m\prime}(\omega)$ are complex-valued coefficients, $j_n(kr)$ and $y_n(kr)$ are the spherical Bessel function of the first and second kind, respectively, and

$$k^2 = -i\omega\sigma\mu. \tag{21}$$

Hereinafter, $k$ is referred to as the induction wave-number. Moreover, $1/\mathrm{Im}(k)$ gives the radial scale length of field attenuation due to EM induction, which is referred to as the skin depth. We can set

$$\beta_n^m(\omega)=0 \text{ and } \beta_n^{m\prime}(\omega)=0 \tag{22}$$

in the present case of EM induction in the Earth with an external inducing field, from the physical requirement for the scalar functions $\chi$ and $\chi'$ to be regular at $r=0$.

Using (20) and (20)′ for the scalar functions $\chi$ and $\chi'$ in (11), the vector potential can be derived as

$$\mathbf{\Pi}(\mathbf{r},\omega) = \sum_{n,m}\begin{pmatrix} 0 \\ \dfrac{1}{\sin\theta}R_n^m\dfrac{\partial Y_n^m}{\partial\varphi} \\ -R_n^m\dfrac{\partial Y_n^m}{\partial\theta}\end{pmatrix} + \sum_{n,m}\begin{pmatrix} \dfrac{1}{r}R_n^{m\prime}\,n(n+1)\,Y_n^m \\ \dfrac{1}{r}\dfrac{\partial(rR_n^{m\prime})}{\partial r}\dfrac{\partial Y_n^m}{\partial\theta} \\ \dfrac{1}{r\sin\theta}\dfrac{\partial(rR_n^{m\prime})}{\partial r}\dfrac{\partial Y_n^m}{\partial\varphi}\end{pmatrix} \tag{23}$$

As a physical requirement, the electric current should not flow across the Earth's surface (at $r=a$), which constrains the coefficient for the toroidal mode as

$$\alpha_n^{m\prime}(\omega)=0. \tag{24}$$

This means that all three components of the second term (the toroidal mode) of the right-hand side of (23) vanish, and we have only to consider the first term (the poloidal mode). Price (1950) proved that the magnetic field of the toroidal mode diminishes outside the conductor, by showing the electric field (vector potential) to be derived by a spatial gradient of a scalar function. This proof is, however, doubtful and meaningless, because the toroidal mode is defined only inside the conductor.

Using the obtained general solution and (7), we derive expressions for the three components of magnetic induction inside the Earth ($r<a$), as follows:

$$B_r(\mathbf{r},\omega)=\frac{1}{r}\sum_{n,m}R_n^m(r,\omega)n(n+1)Y_n^m(\theta,\varphi) \tag{25}$$

$$B_\theta(\mathbf{r},\omega)=\frac{1}{r}\sum_{n,m}\frac{\partial[rR_n^m(r,\omega)]}{\partial r}\frac{\partial Y_n^m(\theta,\varphi)}{\partial\theta} \tag{26}$$

$$B_\varphi(\mathbf{r},\omega)=\frac{1}{r\sin\theta}\sum_{n,m}\frac{\partial[rR_n^m(r,\omega)]}{\partial r}\frac{\partial Y_n^m(\theta,\varphi)}{\partial\varphi}. \tag{27}$$

Applying the boundary condition (continuity of each magnetic component at $r=a$) to (15) through (17) and (25) through (27) results in a set of two equations,

$$(n+1)\iota_n^m(\omega)-n\epsilon_n^m(\omega)=\frac{n(n+1)}{a}R_n^m(a,\omega) \tag{28}$$

and

$$\epsilon_n^m(\omega)+\iota_n^m(\omega)=\frac{-1}{a}\frac{d}{dr}rR_n^m(r,\omega)\Big|_{r=a}, \tag{29}$$

for each spherical harmonic degree and order.

The $Q$-response of the geomagnetic depth sounding (Banks, 1969) is defined as the ratio of internal to external coefficients and is obtained from (28) and (29) as

$$Q_n^m(\omega)=\frac{\iota_n^m(\omega)}{\epsilon_n^m(\omega)}$$
$$=\frac{n}{n+1}\frac{\frac{d}{dr}R_n^m(r,\omega)\Big|_{r=a}-nR_n^m(a,\omega)/a}{\frac{d}{dr}R_n^m(r,\omega)\Big|_{r=a}+(n+1)R_n^m(a,\omega)/a}. \tag{30}$$

Considering condition (22), (30) becomes independent of harmonic order $m$ and may be written as

$$Q_n(\omega)=\frac{\iota_n^m(\omega)}{\epsilon_n^m(\omega)}$$
$$=\frac{n}{n+1}\frac{\frac{d}{dr}j_n(kr)\Big|_{r=a}-nj_n(ka)/a}{\frac{d}{dr}j_n(kr)\Big|_{r=a}+(n+1)j_n(ka)/a}, \tag{31}$$

in the case of a homogeneous conductor. Thus, if we obtain the $Q$-response at a frequency with spherical harmonic expansion, we can estimate the electrical conductivity, $\sigma$, of the Earth. It is possible to derive an analytic solution for $R_n^m(a,\omega)$ independently of the harmonic order $m$ if the Earth's conductivity structure is one-dimensional (1-D), varying only in the radial direction; but, in this case, the $Q$-responses at different frequencies must be obtained in order to determine the structure (e.g., Banks, 1969).

From the solution for the vector potential (23) and (9), we can derive expressions for the three components of the electric field inside the conducting sphere as

$$E_r(\mathbf{r},\omega)=0 \tag{32}$$

$$E_\theta(\mathbf{r},\omega)=-i\omega\sum_{n,m}\frac{1}{\sin\theta}R_n^m(r,\omega)\frac{\partial Y_n^m(\theta,\varphi)}{\partial\varphi} \tag{33}$$

$$E_\varphi(\mathbf{r},\omega)=i\omega\sum_{n,m}R_n^m(r,\omega)\frac{\partial Y_n^m(\theta,\varphi)}{\partial\theta} \tag{34}$$

Using (26), (27), (33), and (34), we have an expression for the impedance at a particular harmonic degree $n$ at the surface ($r=a$) of a homogeneous conducting sphere as

$$Z_{\varphi\theta}{}^{(n)}(\omega)=-Z_{\theta\varphi}{}^{(n)}(\omega)=Z_n(\omega)$$
$$=i\omega\mu\left[\frac{1}{a}+\frac{1}{j_n(ka)}\frac{d}{dr}j_n(kr)\Big|_{r=a}\right]^{-1}. \tag{35}$$

Note again that the impedance does not depend on the harmonic order $m$, when electrical conductivity is laterally uniform.

**2.2 Solution for regional/local induction in a Cartesian coordinate system**

Next, we attempt to solve (6) by variables separation in a Cartesian coordinate system by letting (Price, 1950),

$$\psi(\mathbf{r},\omega)=S_0(x,y)Z_0(z,\omega). \tag{36}$$

Substituting (36) into (6), we have

$$\frac{1}{S_0}\left(\frac{\partial^2 S_0}{\partial x^2}+\frac{\partial^2 S_0}{\partial y^2}\right)=-\frac{1}{Z_0}\frac{\partial^2 Z_0}{\partial z^2}=-\nu^2 \tag{37}$$

where $\nu$ is a real-valued constant hereinafter referred to as the source (inducing field) wavenumber, and we set $\nu>0$ without loss of generality. Then, the horizontal and vertical functions $S_0$ and $Z_0$ are found to satisfy

$$\frac{\partial^2 S_0}{\partial x^2}+\frac{\partial^2 S_0}{\partial y^2}+\nu^2 S_0=0 \tag{38}$$

and

$$\frac{\partial^2 Z_0}{\partial z^2}-\nu^2 Z_0=0, \tag{39}$$

respectively. If we assume a harmonic function for a particular solution of (38)

$$S_0(x,y)\sim e^{i\boldsymbol{\nu}\cdot\mathbf{s}}, \tag{40}$$

where

$$\boldsymbol{\nu}=(\nu_x,\nu_y)^t, \tag{41}$$

$$\mathbf{s}=(x,y)^t, \tag{42}$$

then we will have a particular solution for (39) as

$$Z_0(z,\omega)\sim e^{\pm\nu z}, \tag{43}$$

where

$$\nu^2=\boldsymbol{\nu}^2=\nu_x{}^2+\nu_y{}^2. \tag{44}$$

If we define $\nu_x$ and $\nu_y$ to be unbounded real numbers in $[-\infty,\infty]$, a general solution for (36) is given by a Fourier integral as

$$\psi(\mathbf{r},\omega)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\epsilon_0(\nu,\omega)e^{-\nu z}+\iota_0(\nu,\omega)e^{+\nu z}]e^{i\boldsymbol{\nu}\cdot\mathbf{s}}d\nu_x d\nu_y. \tag{45}$$

Considering the behavior of the right-hand side of (45) for the limit $\nu z\to-\infty$, we find that complex-valued coefficients $\epsilon_0(\nu,\omega)$ and $\iota_0(\nu,\omega)$ correspond to the amplitudes of the externally inducing and internally induced fields, respectively. Using (5) and (45), we derive expressions for the three components of magnetic induction above the surface ($z<0$) of the semi-infinite uniform conductor,

$$B_x(\mathbf{r},\omega)=-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}i\nu_x[\epsilon_0(\nu,\omega)e^{-\nu z}+\iota_0(\nu,\omega)e^{+\nu z}]e^{i\boldsymbol{\nu}\cdot\mathbf{s}}d\nu_x d\nu_y \tag{46}$$

$$B_y(\mathbf{r},\omega)=-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}i\nu_y[\epsilon_0(\nu,\omega)e^{-\nu z}+\iota_0(\nu,\omega)e^{+\nu z}]e^{i\boldsymbol{\nu}\cdot\mathbf{s}}d\nu_x d\nu_y \tag{47}$$

and

$$B_z(\mathbf{r},\omega)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\nu[\epsilon_0(\nu,\omega)e^{-\nu z}-\iota_0(\nu,\omega)e^{+\nu z}]e^{i\boldsymbol{\nu}\cdot\mathbf{s}}d\nu_x d\nu_y. \tag{48}$$

Inside the conductor ($z>0$), we also solve (10) by decomposing the vector potential into poloidal and toroidal modes as given in (12) and write the variable separation solutions of (13) and (13)′ in a Cartesian coordinate system as

$$\chi(\mathbf{r},\omega)=S_1(x,y)Z_1(z,\omega), \tag{49}$$

and

$$\chi'(\mathbf{r},\omega)=S_1'(x,y)Z_1'(z,\omega). \tag{49$'$}$$

Substituting (49) and (49)′ into (13) and (13)′, respectively, we have

$$\frac{1}{S_1}\left(\frac{\partial^2 S_1}{\partial x^2}+\frac{\partial^2 S_1}{\partial y^2}\right)=\kappa^2-\frac{1}{Z_1}\frac{\partial^2 Z_1}{\partial z^2}=-\nu^2 \tag{50}$$

and

$$\frac{1}{S_1'}\left(\frac{\partial^2 S_1'}{\partial x^2}+\frac{\partial^2 S_1'}{\partial y^2}\right)=\kappa^2-\frac{1}{Z_1}\frac{\partial Z_1'}{\partial z^2}=-\nu^2, \tag{50$'$}$$

where

$$\kappa^2=i\omega\sigma\mu. \tag{51}$$

Here, $\kappa$ is the induction wavenumber. Again, $1/\mathrm{Re}(\kappa)$ gives the vertical length scale of field attenuation due to EM induction, which is referred to as the skin depth. The non-dimensional ratio of induction to source wavenumbers $\mathrm{Re}(\kappa)/\nu$ is referred to as the induction number (e.g., Utada and Munekane, 2000).

The horizontal and vertical functions, $S_1$, $S_1'$, $Z_1$, and $Z_1'$ can be obtained as solutions of

$$\frac{\partial^2 S_1}{\partial x^2}+\frac{\partial^2 S_1}{\partial y^2}+\nu^2 S_1=0, \tag{52}$$

$$\frac{\partial^2 S_1'}{\partial x^2}+\frac{\partial^2 S_1'}{\partial y^2}+\nu^2 S_1'=0, \tag{52$'$}$$

$$\frac{\partial^2 Z_1}{\partial z^2}-(\nu^2+\kappa^2)Z_1=0, \tag{53}$$

and

$$\frac{\partial Z_1'}{\partial z^2}-(\nu^2+\kappa^2)Z_1'=0, \tag{53$'$}$$

respectively. As in the case outside the conductor, as shown above, we assume the horizontal functions in (52) and (52)′ both to be harmonic functions, so,

$$S_1(x,y),\ S_1'(x,y)\sim e^{i\boldsymbol{\nu}\cdot\mathbf{s}}. \tag{54}$$

Then, particular solutions for (53) and (53)′ can be written as

$$Z_1(z,\omega),\ Z_1'(z,\omega)\sim e^{\pm\gamma z}, \tag{55}$$

where

$$\gamma^2=\nu^2+\kappa^2. \tag{56}$$

Here, $\mathrm{Re}(\gamma)$ gives the vertical length scale of non-plane field attenuation. The general solutions for (49) and (49)′ can be written in the form of a Fourier integral, as follows:

$$\chi(\mathbf{r},\omega)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\epsilon_1(\nu,\omega)e^{-\gamma z}+\iota_1(\nu,\omega)e^{+\gamma z}]e^{i\nu\cdot s}d\nu_x d\nu_y \tag{57}$$

and

$$\chi'(\mathbf{r},\omega)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\epsilon_1'(\nu,\omega)e^{-\gamma z}+\iota_1'(\nu,\omega)e^{+\gamma z}]e^{i\nu\cdot s}d\nu_x d\nu_y. \tag{57$'$}$$

Then, we can set,

$$\iota_1(\omega)=0 \text{ and } \iota_1'(\omega)=0 \tag{58}$$

by considering the behavior of $Z_1$ to be regular at $\mathrm{Re}\,(\gamma z)\to\infty$ inside the uniform conductor. Finally, we obtain an expression for vector potential consisting of poloidal and toroidal modes as

$$\mathbf{\Pi}(\mathbf{r},\omega)=\nabla\times\begin{pmatrix}0\\0\\S_1Z_1\end{pmatrix}+\nabla\times\nabla\times\begin{pmatrix}0\\0\\S_1'Z_1'\end{pmatrix}=\begin{pmatrix}\Pi_x\\\Pi_y\\\Pi_z\end{pmatrix}+\begin{pmatrix}\Pi_x'\\\Pi_y'\\\Pi_z'\end{pmatrix}, \tag{59}$$

where

$$\Pi_x(\mathbf{r},\omega)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}i\nu_y[\epsilon_1(\nu,\omega)e^{-\gamma z}]e^{i\nu\cdot s}d\nu_x d\nu_y, \tag{60}$$

$$\Pi_y(\mathbf{r},\omega)=-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}i\nu_x[\epsilon_1(\nu,\omega)e^{-\gamma z}]e^{i\nu\cdot s}d\nu_x d\nu_y, \tag{61}$$

$$\Pi_z(\mathbf{r},\omega)=0, \tag{62}$$

and

$$\Pi_x'(\mathbf{r},\omega)=-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}i\nu_x\gamma[\epsilon_1'(\nu,\omega)e^{-\gamma z}]e^{i\nu\cdot s}d\nu_x d\nu_y, \tag{60$'$}$$

$$\Pi_y'(\mathbf{r},\omega)=-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}i\nu_y\gamma[\epsilon_1'(\nu,\omega)e^{-\gamma z}]e^{i\nu\cdot s}d\nu_x d\nu_y, \tag{61$'$}$$

$$\Pi_z'(\mathbf{r},\omega)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\nu^2[\epsilon_1'(\nu,\omega)e^{-\gamma z}]e^{i\nu\cdot s}d\nu_x d\nu_y. \tag{62$'$}$$

The condition whereby the electric current is not allowed to flow across the surface ($z=0$) in (59) requires the vertical function (62)′ to diminish, i.e., $\epsilon_1'=0$, which leads both horizontal functions (60)′ and (61)′ also to diminish everywhere inside the conductor ($z>0$). Again, we have only to consider the poloidal mode inside the conductor, as in the case of spherical coordinate system solutions. Substituting (60) through (62) into (7), we obtain expressions for the three components of the magnetic induction inside the conductor as

$$B_x(\mathbf{r},\omega)=-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}i\nu_x\gamma[\epsilon_1(\nu,\omega)e^{-\gamma z}]e^{i\nu\cdot s}d\nu_x d\nu_y \tag{63}$$

$$B_y(\mathbf{r},\omega)=-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}i\nu_y\gamma[\epsilon_1(\nu,\omega)e^{-\gamma z}]e^{i\nu\cdot s}d\nu_x d\nu_y \tag{64}$$

$$B_z(\mathbf{r},\omega)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\nu^2[\epsilon_1(\nu,\omega)e^{-\gamma z}]e^{i\nu\cdot s}d\nu_x d\nu_y. \tag{65}$$

If we apply the boundary condition (continuation of each of the three components) at the surface of the conductor ($z=0$), we obtain a set of two equations relating coefficients outside and inside the conductor,

$$\epsilon_0(\nu,\omega)+\iota_0(\nu,\omega)=\gamma\epsilon_1(\nu,\omega), \tag{66}$$

$$\epsilon_0(\nu,\omega)-\iota_0(\nu,\omega)=\nu\epsilon_1(\nu,\omega). \tag{67}$$

Then, the ratio of external to internal coefficients can be derived as

$$q(\nu,\omega)=\frac{\iota_0(\omega)}{\epsilon_0(\omega)}=\frac{\gamma-\nu}{\gamma+\nu}, \tag{68}$$

which is a response function similar to $Q_n$ in a global approach (31). This result indicates that we can estimate electrical conductivity using (68), if the source wavenumber $\nu$ is accurately estimated with an array observation.

From (60) through (62) and (9), expressions for the three components of the electric field can be derived as

$$E_x(\mathbf{r},\omega)=-i\omega\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}i\nu_y[\epsilon_1(\nu,\omega)e^{-\gamma z}]e^{i\nu\cdot s}d\nu_x d\nu_y, \tag{69}$$

$$E_y(\mathbf{r},\omega)=i\omega\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}i\nu_x[\epsilon_1(\nu,\omega)e^{-\gamma z}]e^{i\nu\cdot s}d\nu_x d\nu_y, \tag{70}$$

and

$$E_z(\mathbf{r},\omega)=0. \tag{71}$$

Using (63), (64), (69), and (70), we obtain an expression for the impedance of each source wavenumber as

$$Z_{xy}(\nu,\omega)=-Z_{yx}(\nu,\omega)=\frac{i\omega\mu}{\gamma}. \tag{72}$$

## 3. Plane-wave approximation

The term 'plane-wave' is usually defined as a wave at a point sufficiently distant (as compared to wavelength) from the radiation source. When used in EM induction problems in which the basic equation differs from a wave equation, an alternative definition appropriate to its physical meaning is needed. Price (1950) proposed an expression for the scalar potential of a spatially uniform field in the form of (36) with the same $S_0$, but with

$$Z_0(z,\omega)=A_0(\omega)z+B_0(\omega) \tag{73}$$

which is a general solution of (39) in the case of $\nu=0$ ($A_0$ and $B_0$ are constants). If we compare (73) with a general solution for the case of finite $\nu$ (45) by letting $\nu=0$ in (45), we find the simple relation

$$B_0(\omega)=\epsilon_0(0,\omega)+\iota_0(0,\omega), \tag{74}$$

and that $A_0$ is independent. If we apply boundary conditions at the surface after deriving the three components of the magnetic field and using the internal solution given by (63) through (65) with $\nu=0$, we have

$$A_0(\omega)=0. \tag{75}$$

This means that the spatially uniform inducing field with a term $A_0(\omega)z$ in the scalar potential makes no contribution to EM induction in a semi-infinite conductor with a flat surface. Although Price (1950) proved that the problem becomes indeterminate using the scalar potential of this form, we find that the inclusion of $A_0(\omega)z$ in (73) also has no mathematical significance, because the form of (43) does not uniformly converge to the form of (73) at a limit of $\nu\to 0$.

Price (1962) argued the indeterminate problem again, by letting $\nu=0$ in (43) this time and $S_0(x,y)=1$ in (36). However, the latter seems to be either a typo or a simple mistake by the author, because the horizontal gradients of $S_0$ must be finite in order to produce horizontal components of finite intensity. Among the possible particular solutions of

$$\frac{\partial^2 S_0}{\partial x^2}+\frac{\partial^2 S_0}{\partial y^2}=0, \tag{76}$$

which is a special case of (38) for $\nu=0$, we may choose a linear function

$$S_0(x,y)=a(\omega)x+b(\omega)y+c(\omega) \tag{77}$$

for the present case, because it produces laterally uniform horizontal magnetic components to have finite (bounded) intensity at infinity. Substituting (77) into (36) and (5) yields

$$B_x(\mathbf{r},\omega)=-a(\omega)[\epsilon_0(0,\omega)+\iota_0(0,\omega)], \tag{78}$$

$$B_y(\mathbf{r},\omega)=-b(\omega)[\epsilon_0(0,\omega)+\iota_0(0,\omega)], \tag{79}$$

and

$$B_z(\mathbf{r},\omega)=0. \tag{80}$$

If we apply boundary conditions at the surface, we have only

$$\epsilon_0(0,\omega)+\iota_0(0,\omega)=\kappa\epsilon_1(0,\omega). \tag{81}$$

In this case, the problem of determining induced fields inside and outside the conductor ($\iota_0$ and $\epsilon_1$) when an external inducing field ($\epsilon_0$) is given becomes indeterminate, as suggested by Price (1962).

This consequence is not surprising, because we let the vertical components inside and outside the conductor diminish automatically by letting $\nu=0$ in the form of the scalar potential (36). The magnetic field due to the potential of this form is absolutely uniform, which does not exist in reality. Now, we realize that the treatment of the inducing field as a plane-wave (spatially uniform field) is an approximation of a special case of EM induction, in which the spatial non-uniformity of the inducing field can be neglected. Such a situation is simply expressed as

$$\nu\ll|\kappa|. \tag{82}$$

If we apply the condition of the plane-wave approximation of (82) to (66) and (67), we have a set of two equations relating the three coefficients,

$$\epsilon_0(\nu,\omega)+\iota_0(\nu,\omega)=\kappa\epsilon_1(\nu,\omega), \tag{83}$$

and

$$\epsilon_0(\nu,\omega)-\iota_0(\nu,\omega)=\nu\epsilon_1(\nu,\omega), \tag{84}$$

and therefore the problem is determinate.

If the condition for a plane-wave approximation (82) holds, the response function defined by (68) approaches unity,

$$q(\nu,\omega)=\frac{\iota_0(\nu,\omega)}{\epsilon_0(\nu,\omega)}=\frac{\gamma-\nu}{\gamma+\nu}\to 1-2\frac{\nu}{\kappa}\to 1, \tag{85}$$

and the vertical component of the magnetic field vanishes, which also shows that the total (inducing plus induced) field is nearly horizontal and is almost twice as intense as the inducing field at the surface. In this way, the total magnetic field appears to be planar at the surface of a semi-infinite conductor under the condition given by (82), as if it resulted from plane-wave incidence. Thus, it is a physically reasonable condition for a plane-wave approximation of EM induction.

The condition for a plane-wave approximation obtained here is also consistent with the basic theory of the MT method (Cagniard, 1953), and we can derive an expression for the impedance from (72) as

$$Z_{xy}(\nu,\omega)=-Z_{yx}(\nu,\omega)=\frac{i\omega\mu}{\gamma}\rightarrow\frac{i\omega\mu}{\kappa}, \tag{86}$$

at a plane-wave approximation ($\nu\ll|\kappa|$).

Next, we attempt to apply the concept of a plane-wave approximation to the case of the global induction approach in a spherical coordinate system. To do this, we need to extend the condition (82) to the case of a spherical conductor. If we apply spherical harmonic expansion (18), the spatial wavenumber of the inducing field of the spherical harmonic degree $n$ is roughly estimated (Srivastava, 1966; actual derivation is given in the next section) as

$$\nu\sim\frac{n}{a} \tag{87}$$

and therefore the condition can be written as

$$\frac{n}{a}\ll|k|. \tag{88}$$

Weaver (1994) suggested that the expansion coefficients at smaller spherical harmonic degrees are more appropriate for approximating a spatially uniform inducing field (plane-wave). Now, we understand that this condition (low harmonic degree) describes a part of the condition (88) when $|k|$ (or the frequency) is fixed.

The asymptotic behavior of the spherical Bessel function

$$\frac{1}{j_n(ka)}\frac{d}{dr}j_n(kr)\bigg|_{r=a}\rightarrow ik\ \text{ for }\ |ka|\rightarrow\infty, \tag{89}$$

helps us to understand the behaviors of the response functions discussed in Section 2.1. For example, substituting (89) into (31) gives the expression for $Q$-response with a plane-wave approximation ( $\frac{n}{a}\ll|k|$ ) as

$$Q_n(\omega)=\frac{\iota_n(\omega)}{\epsilon_n(\omega)}\rightarrow\frac{n}{n+1}\frac{ik-n/a}{ik+(n+1)/a}\rightarrow\frac{n}{n+1}. \tag{90}$$

If we consider a degree $n$ term, the right-hand side of the spherical harmonic expansion of $B_r$ (15) at the surface ($r=a$) becomes

$$n\epsilon_n^m(\omega)-(n+1)\iota_n^m(\omega)\rightarrow 0, \tag{91}$$

which means that the radial component of the magnetic field diminishes. In other words, the magnetic lines at a plane-wave approximation become parallel to the surface of a spherical conductor, and are not spatially uniform.

On the other hand, the asymptotic behavior of impedance (35) with a plane-wave approximation is derived as

$$Z_n(\omega)=i\omega\mu\left[\frac{1}{a}+\frac{1}{j_n(ka)}\frac{d}{dr}j_n(kr)\bigg|_{r=a}\right]^{-1}\rightarrow\frac{\omega\mu}{k}, \tag{92}$$

in a spherical coordinate system. Considering the identity $\kappa=ik$, the results of (90) and (92) are consistent with those of (85) and (86) with a plane-wave approximation in a Cartesian coordinate system, respectively. Schmucker (1985) derived the same result, which he referred to as the zero-wavenumber approximation.

The geomagnetic transfer function is expressed as the ratio of the radial to azimuthal components ($B_r/B_\theta$ or $B_r/B_\varphi$) of the magnetic field variations at long periods to explore deep structures, assuming a ring current ($n=1$, $m=0$) source (e.g., Banks, 1969). Expressions (63) through (65) and (69) through (70) indicate that the corresponding transfer function in a Cartesian coordinate system ($B_z/B_x$ or $B_z/B_y$) for a particular value of $\nu$ is $O(\nu/\gamma)$, while the impedance is $O(1/\gamma)$. This proportionality is consistent with the simple relation between impedance and geomagnetic transfer function: the latter is approximated by a lateral gradient of the former (Utada and Munekane, 2000). Therefore, the source field effect in the geomagnetic transfer function tends to be more intense than that in the impedance (Jones and Spratt, 2002; Shimizu *et al.*, 2011; Murphy and Egbert, 2018) if they are estimated from the same set of EM data. The proportionality also suggests that the transfer function should be smaller at shorter periods at the surface of a uniform (or 1-D) conductor. However, observation results at short periods often exhibit transfer functions of considerable amplitudes. This is because lateral heterogeneity causes a significant secondary magnetic field with a much higher wavenumber than the source and induction wavenumbers. Accordingly, the ratio of vertical to horizontal components of the magnetic field variations is used as a response function that reflects the lateral conductivity contrast in regional/local induction studies at short periods, known as the induction vector or the tipper (Chave and Jones, 2012).

## 4. Flat Earth approximation

Here, we consider the case in which the Earth can be regarded as a semi-infinite conductor with a plane surface. This fundamental condition allows solutions of the EM induction equation to be used in a Cartesian coordinate system. Thus far, we have arbitrarily treated global/semi-global and regional/local approaches with formulations in spherical and Cartesian coordinate systems, respectively. The previous section shows the equivalence of corresponding solutions using different approaches with a plane-wave approximation. However, at a certain limit of the global approach, where a flat Earth approximation is allowed, corresponding solutions in a spherical coordinate system should not be distinguishable from those in a Cartesian coordinate system, regardless of whether a plane-wave approximation holds. Honkura and Rikitake (1985) and Weaver (1994) argued this problem independently, and they derived different results. First, we re-examine the results reported by these authors.

Honkura and Rikitake (1985) suggested that the problem of EM induction in a semi-infinite conductor (a flat Earth approximation) due to an inducing field having an infinite wavelength (a plane-wave approximation) may be treated by considering the case $a \to \infty$ in EM induction theory for a spherical conductor, where $a$ is the Earth's radius. They further argued that they obtained $Q_n(\omega) \to \frac{n}{n+1}$ in this case, which is consistent with (90) in a plane-wave approximation. The authors then pointed out that the problem of estimating $Q_n(\omega)$ becomes indeterminate, because its value can take any value between 1/2 and 1 with harmonic degree $n$ being any positive integer. The Earth's surface can be regarded as completely flat if the Earth's radius is infinitely large. However, such a mathematical treatment ($a \to \infty$) is not appropriate, as the Earth's radius is not a variable. Therefore, this condition must be rephrased with (88), which is actually the condition for a plane-wave approximation. However, we need a condition for a flat Earth approximation for a case in which a plane-wave approximation does not hold, in order to argue the consistency of $Q$-responses estimated by different approaches. We need an additional derivation for this problem.

Weaver (1994) examined the consistency between two estimates of the $Q$-response with a plane-wave approximation: one estimate is $Q_n(\omega)$ from a global approach under a condition in which a flat Earth approximation is allowed, and the other estimate is $q(\nu,\omega)$ from a regional approach (68). Then, the author also took a limit of $a \to \infty$ to have the value of the $Q$-response under a flat Earth approximation (which is inappropriate, as we have already pointed out) and derived $Q_n(\omega) \to \frac{n}{n+1}$. For this result, he suggested taking the smallest value of harmonic degree, i.e., $n=1$, for a plane-wave approximation, because this value corresponds to a nearly uniform inducing field. Thus, he demonstrated that the value of $Q$-response in a global approach satisfying both conditions for plane-wave and for flat Earth approximations converges to 1/2. However, as we have already shown in (85), the expression for the response function $q(\nu,\omega)$ approaches unity at the limit of a plane-wave approximation. The author concluded the non-uniqueness of the response estimates using two different approaches, although they should not be distinguishable at an appropriate limit.

In order to solve these apparent contradictions, we first need to revise the condition for a flat Earth approximation instead of letting $a \to \infty$. Here, we attempt to obtain an appropriate condition by considering a case in which the basic equation for the radial function of (19) is approximated by that for the vertical function (53) in the variable separation solutions (Wait, 1962; Srivastava, 1966). If we consider a small area of the Earth's surface ($r=a$), the radial direction can be replaced by the vertical direction ($r-a=-z$), so that the radial derivative in a spherical coordinate system can be replaced by the vertical derivative in a Cartesian coordinate system ($\frac{d}{dr}=-\frac{d}{dz}$). Note that such a consideration is allowed only when we ignore lateral variations of electrical conductivity. Then, the radial function $R_n^m(r)$ in (19) can be regarded as a function of the vertical coordinate $z$, which satisfies a differential equation in a local Cartesian coordinate system,

$$\frac{d^2R_n^m(z)}{dz^2}-\frac{2}{a-z}\frac{dR_n^m(z)}{dz}+\left\{k^2-\frac{n(n+1)}{(a-z)^2}\right\}R_n^m(z)=0. \tag{93}$$

When addressing the EM induction problem in a Cartesian coordinate system, we consider a depth range, which is much smaller than the Earth's radius, and therefore $a-z$ in (93) is approximated by $a$ for the estimation of the order of magnitude. Thus, (93) is further approximated as

$$\frac{d^2R_n^m(z)}{dz^2}-\frac{2}{a}\frac{dR_n^m(z)}{dz}+\left\{k^2-\frac{n(n+1)}{a^2}\right\}R_n^m(z)=0. \tag{94}$$

We rewrite the coefficient of the third term of (94) as

$$\gamma_n{}^2=\nu_n{}^2-k^2=\nu_n{}^2+\kappa^2, \tag{95}$$

where $\gamma_n$ can be regarded as the total wavenumber representing the spatial variation of the EM field in the Earth, and

$$\nu_n=\frac{\sqrt{n(n+1)}}{a} \tag{96}$$

is an expression for the spatial wavenumber of the inducing field in (56) in terms of the spherical harmonic degree $n$. Note that Srivastava (1966) suggested that the spatial wavenumber $\nu$ of the inducing field with a flat Earth approximation can be related to the spherical harmonic degree $n$ as $\nu\sim\frac{n}{a}$, as shown in the previous section, which turned out to correspond to (96) in the case of large $n$.

If the second term of (94) is negligible, the differential equation is essentially identical to (53), the particular solution of which takes the following form:

$$R_n^m(z,\omega)\sim e^{-\gamma_n z}. \tag{97}$$

With this solution, the second term of (94) is scaled to be $O(|\gamma_n/a|)$, while the first and third terms are scaled to be $O(|\gamma_n|^2)$. Thus, we derive a condition for the second term of (94) to be of a negligible order as

$$|\gamma_n a|\gg 1, \tag{98}$$

which is regarded as the condition for a flat Earth approximation.

Because the spatial wavenumber $\gamma_n$ consists of two terms, $\nu_n$ and $k$, as shown in (95), we need to consider two cases separately: the case in which the wavenumber of the inducing field dominates, $\nu_n>|k|$ (a diffusion-dominant case), and the case in which the induction wavenumber dominates the total wavenumber $\gamma_n$, i.e., $\nu_n<|k|$, which can be called an induction-dominant case. Thus, we have two conditions for a flat Earth approximation in diffusion- and induction-dominant cases as

$$n\gg 1 \tag{99}$$

or

$$\nu_n\ll|k|, \tag{100}$$

respectively.

The condition (100) is essentially that for a plane-wave approximation. This means that, if a plane-wave approximation is allowed, a flat Earth approximation is always allowed. On the other hand, if the harmonic degree is a large number, then (99) is the condition for a flat Earth approximation when a plane-wave approximation does not hold. This can be a common condition for a flat Earth approximation in other geophysical methods, such as gravity or geomagnetism for a scalar potential. If either (99) or (100) is satisfied, the radial derivative in the $Q$-response (31) can be replaced by the vertical derivative, which is scaled by $-\gamma_n$, and therefore we have

$$Q_n(\omega)\sim\frac{n}{n+1}\frac{\gamma_n a-n}{\gamma_n a+(n+1)} \tag{101}$$

for a large value of the harmonic degree $n$ that allows a flat Earth approximation. Then, for a large value of harmonic degree $n$, the response (101) approaches unity,

$$Q_n(\omega)\rightarrow\frac{n}{n+1}\rightarrow 1 \tag{102}$$

with a plane-wave approximation ($\nu_n\ll|k|$), which is consistent with $q(\nu,\omega)\rightarrow 1$ in a Cartesian coordinate system with $\nu\ll|\kappa|$ (85).

More precisely, we may have both induction-dominant and diffusion-dominant cases for a single observation. If the inducing field wavenumber has only slight frequency dependence, the induction number will be dependent on the square-root of the frequency as

$$\frac{\mathrm{Im}(k)}{\nu_n}=\frac{1}{\sqrt{2}}\frac{|k|}{\nu_n}\propto\sqrt{\omega}. \tag{103}$$

Therefore, roughly, we can expect an induction-dominant case at high frequencies and a diffusion-dominant case at low frequencies in such a situation. However, note that the frequency dependence is opposite when we consider Sq harmonics. As shown by Schmucker (1999), the principal components of the Sq signal have mode frequencies $\omega_p$ expressed using the mode number $p$ as

$$\omega_p = p\omega_0 \quad (p=1, 2, 3...), \tag{104}$$

where $\omega_0$ is the fundamental frequency of Sq (corresponding to the period of one day), while the harmonic degree and order of the corresponding mode are given by

$$n_p = p+1 \text{ and } m_p = p, \tag{105}$$

respectively. Using (21) and (96), we obtain expressions for corresponding wavenumbers of EM induction and the inducing field with a flat Earth approximation as

$$k_p{}^2 = -i\omega_p\sigma\mu = -ip\omega_0\sigma\mu, \tag{106}$$

and

$$\nu_p = \frac{\sqrt{n_p(n_p+1)}}{a} = \frac{\sqrt{(p+1)(p+2)}}{a}. \tag{107}$$

The last equation indicates that the inducing field wavenumber has a strong mode number dependence and, therefore, frequency dependence. Thus, we obtain the dependence of the induction number on the corresponding mode number or mode frequency as

$$\frac{\mathrm{Im}(k_p)}{\nu_p} = \frac{1}{\sqrt{2}}\frac{|k_p|}{\nu_p} \propto \frac{1}{\sqrt{p}} \propto \frac{1}{\sqrt{\omega_p}}, \tag{108}$$

which means that the source field effect tends to be more effective for higher harmonics with higher frequencies (Shimizu *et al.*, 2011).

## 5. Conclusions and additional notes

We have reviewed previous studies on the conditions for two main approximations in the EM induction method, plane-wave approximation and flat Earth approximation, based on simple analytic solutions for a uniform conductor both in spherical and Cartesian coordinate systems. The results show that the magnetic field becomes parallel to the surface of the conductor when the condition for the plane-wave approximation is satisfied. It is also demonstrated that the solutions in a spherical coordinate system become equivalent to those in a Cartesian coordinate system when the condition for the flat Earth approximation is satisfied. In addition, we show that the condition for the flat Earth approximation is automatically satisfied when the plane-wave approximation condition is satisfied in general. If calculations in both coordinate systems provide equivalent results, we generally prefer the calculation in a Cartesian coordinate system, which is easier. This is the case that most MT studies examine. However, note that a map projection, $(\theta, \varphi) \rightarrow (x, y)$, is necessary for the treatment in a Cartesian coordinate system, and that no map projection can preserve shape and size simultaneously. A projection distortion of shape, size, and azimuth at peripheral regions is inevitable. This is not problematic as long as we consider a 1-D conductor such as that considered in the present review paper or the observation array covers only a local area (typical scale ≤ 100 km). However, this may cause a serious problem when a MT study is performed using an array consisting of a number of observation sites distributed over a wide region to reveal an actual (3-D) laterally heterogeneous conductivity distribution (e.g., Miensopust, 2017). In such a study, 3-D MT numerical modeling and inversion are carried out for data interpretation over an area at least several times greater than the array size, in order to account primarily for the effects of distant and large-scale lateral contrasts such as coastlines. In order to avoid peripheral distortion due to a map projection, one possible solution is to make a formulation of the forward part of the inversion problem in a spherical coordinate system, which is a direct approach that should be examined. If inversion must be carried out in a Cartesian coordinate system, careful examination is necessary to confirm that the effects of peripheral distortion are not significant before performing modeling and inversion.

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**References**

Backus, G., 1986. Poloidal and toroidal fields in geomagnetic field modeling. *Rev. Geophys.*, **24**, 75–109.

Banks, R. J., 1969. Geomagnetic variations and the electrical conductivity of the upper mantle. *Geophys. J. Roy. Astr. Soc.*, **17**, 457–487.

Cagniard, L., 1953. Basic theory of the magneto-telluric method of geophysical prospecting. *Geophysics*, **18**, 605–635.

Chave, A.D., Jones, A.G. (ed.), 2012. *The magnetotelluric method*. Cambridge University Press, Cambridge, p.552.

Honkura, Y., Rikitake, T., 1985. *Solid Earth Geomagnetism*. Terra Sci. Pub., Tokyo, pp.384.

Jones, A.G., Spratt, J., 2002. A simple method for deriving the uniform field MT responses in auroral zones. *Earth Planets Space*, **53**, 443–450.

Koch, S., Kuvshinov, A., 2015. 3-D EM inversion of ground based geomagnetic Sq data. Results from the analysis of Australian array (AWAGS) data. *Geophys. J. Int.*, **200**, 1284–1296.

Langel, R.A., 1987. The main field. In "*Geomagnetism*" edited by Jacobs, J.A., vol. I. Academic Press, London, UK, 249–512.

Lezaeta, P., Chave, A.D., Jones, A.G., Evans, R., 2007. Source field effects in the auroral zone: Evidence from Slave Craton (NW Canada). *Phys. Earth Planet. Int.*, **164**, 21–35.

Miensopust, M. P., 2017. Application of 3-D Electromagnetic Inversion in Practice: Challenges, Pitfalls and Solution Approaches, *Surv. Geophys.*, **38**, 869–933.

Murphy, B.S., Egbert, G.D., 2018. Source biases in mid latitude magnetotelluric transfer functions due to Pc3-4 geomagnetic pulsations. *Earth Planets Space*, **70**, doi:10.1186/s40623-018-0781-0.

Price, A.T., 1950. Electromagnetic induction in a semi-infinite conductor with a plane boundary. *Quart. J. Mech. Appl. Math.*, **2**, 283–410.

Price, A.T., 1962. The theory of magneto-telluric methods when the source field is considered. *J. Geophys. Res.*, **67**, 1907–1918.

Pulkkinen, A., BEAR Working Group, 2003. Ionospheric equivalent current distributions determined with the method of spherical elementary current systems. *J. Geophys. Res.*, **108**, doi:10.1029/2001JA005085.

Schmucker, U., 1987. Substitute conductors for electromagnetic response estimates. *PAGEOPH*, **125**, 341–367.

Schmucker, U. 1999. A spherical harmonic analysis of solar daily variations in the years 1964–1965: Response estimates and source fields for global induction - I. Methods. *Geophys. J. Int.*, **136**, 439–454.

Shimizu, H., Yoneda, A., Baba, K., Utada, H., Palshin, N.A., 2011. Sq effect on the electromagnetic response functions in the period range between $10^4$ and $10^5$ s. *Geophys. J. Int.*, **186**, 193–206.

Srivastava, S. P., 1965. Method of interpretation of magnetotelluric data when the source field is considered. *J. Geophys. Res.*, **70**, 945–954.

Srivastava, S.P., 1966. Theory of the magnetotelluric method for a spherical conductor. *Geophys. J. Roy. astr. Soc.*, **11**, 373–387.

Towle, J.N., 1974. Comments on "the theory of magnetotelluric methods when the source field is considered". *J. Geophys. Res.*, **79**, 2143–2143.

Utada, H., Munekane, H., 2000. On galvanic distortion of regional three-dimensional impedances. *Geophys. J. Int.*, **140**, 385–398.

Utada, H., Koyama, T., Shimizu, H., Chave, A.D., 2003. A semi-global reference model for electrical conductivity in the mid mantle beneath the north Pacific region. *Geophys. Res. Lett.*, **30**, No. 4, doi:10.1029/2002GL016092.

Viljanen, A., Pirjola, R., Amm, O., 1999. Magnetotelluric source effect due to 3D ionospheric current system using complex image method for 1D conductivity structures. *Earth Planets Space*, **51**, 933–945.

Wait, J.R., 1954. On the relation between telluric currents and the Earth's magnetic field, *Geophysics*, **19**, 281–289.

Wait, J.R., 1962. Theory of magneto-telluric fields. *J. Res. natn. Bur. Stand.*, D., **66**, 509–541.

Weaver, J.T., 1994. *Mathematical methods for Geo-electromagnetic Induction.* John Wiley & Sons Inc., New York, 316 pp.

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# 自然電磁場変動を用いた電磁誘導法における平面波近似と平板地球近似について

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**要　旨**

通常マグネトテルリク法をはじめとする電磁誘導法では，(1) 地球を半無限導体とみなし，平坦な地表より上方から与えられる (2) 空間的に一様な電磁場変動が起こす電磁誘導を扱うことが多い．(1) と (2) はそれぞれ，空間的に非一様な電磁場変動が，球体の地球内部に生じる電磁誘導の近似的取り扱いである．電磁誘導法の理論的枠組みが確立してから長い年月が経過しているが，この２つの近似の成り立つ物理的条件は系統的に理解されているとは言いがたく，文献には混乱も見られる．小論では，球座標系および直交座標系における電磁誘導問題の基本式を再吟味することにより，２つの近似の条件がどのように導かれているのかを明らかにすることを目指す．得られた結果は，それぞれの座標系での解には適切な極限において必ず整合性があることを示した．すなわち，近似の条件を適切に与えさえすれば，過去の文献で指摘されたような「不定性」や「非一意性」などの問題は生じないことがわかった．

**キーワード：** 電磁誘導，マグネトテルリク法，平面波近似，平板地球近似，電気伝導度