# Research report for the ERI visit from July $1^{st}$ to October $31^{th}$

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## 1 Introduction

I came to ERI as a postdoc fellow between July $1^{st}$ and October $31^{th}$ in order to investigate on the perturbation of TEC (Total Electron Content) generated by tsunamis with Professor Shingo Watada. In order to model this perturbation, we introduce the ionospheric normal modes calculation which is based on the generalized spherical harmonics. We applied our novel technique to reconstruct the sea-level and ionospheric signature of the Haida Gwaii events extracted from real data (DART and GPS, respectively). The good agreement between modeling and observation demonstrates that the normal mode approach is able to reconstruct the waveform of the first arrivals of the tsunami associated with the direct propagation path in the ocean.

## 2 Background

It is known that natural hazard events such as earthquakes, tsunamis and volcano eruptions can generate atmospheric and ionospheric perturbations [11, 19]. During earthquakes, vertical displacements of the ground or of the ocean floor generate pressure waves in the neutral atmosphere that propagate upward and grow in amplitude by several orders of magnitude as they reach ionospheric heights, since the atmospheric density decreases exponentially with height. Then, ionospheric perturbations are formed via the collisions between neutral and charged particles. Such co-seismic ionosphere disturbances (CSID) have been often detected by the Global Navigation Satellite Systems (GNSS) and have been studied in details during the last decades [4, 2, 1]. Unlike earthquakes, propagating tsunamis are known to generate lower frequency gravity waves that further propagate obliquely in the atmosphere and reach the ionospheric heights in 45-60 min [3, 23, 17, 12, 5].

These observations were followed by many modeling efforts. [18, 17] modeled the IGWs using a 2D tsunami wavefield input and neglecting the compressibility of the atmosphere and the attenuation process which occurs in the high atmosphere. Therefore, even if waveform obtained was satisfying, the neglected phenomena leads to a too high IGWs amplitude about 600 m/s higher than the sound speed. [10] and [15] took this attenuation effect in account which reduced significantly the amplitude of the IGWs. This demonstrated that if one aims to model the IGWs in the atmopshere, taking into account the attenuation effect is a must. These first modeling results were rapidly followed by more comprehensive wave propagation models taking into account electromagnetic field perturbations, viscosity and compressibility [12, 13]. Other numerical approaches based on fully non-linear modeling of thermospheric coupling effects [16], the wave packets modeling technique [24] or the perturbation theory of acoustic-gravity waves [8] were also developed.

[5, 22] modeled the TEC perturbation induced by tsunamis using the normal modes summation technique. The normal modes modeling allowed to model the perturbation induced by the tsunami not only in the ocean but also in the solid Earth and the atmosphere. This modeling takes into account the attenuation process of the solid Earth and the compressibility and the viscosity effect in the atmosphere. The summation of tsunami normal modes was able to model TEC data quite accurately, with a 20 % error when modeling the amplitude waveform using a finite source model of the Tohoku earthquake source.

In this paper, in the case of the 2012 Haida Gwaii tsunami, after recalling the basis on the basis of the normal modes and the generalized spherical harmonics theory [20], we introduce the computation method of the ionospheric normal modes (TEC perturbation). At last, we validate this modeling with the good agreement obtained between the GPS data measurements and the modeling.

# 3 Computation of the normal modes

The modes are computed in the whole solid Earth-ocean system. The tsunami branch is one branch of the spheroidal normal modes and can be seen as gravity waves of the oceanic layer. In this work, we computed the normal modes following [14, 5, 22] between $\ell = 40$ and $\ell = 520$, which corresponds to the 0.2 mHz to 2.6 mHz frequency range. The Earth structure internal model is given by is provided by the Preliminary Reference Earth Model (PREM) [7]. The ocean depth is a constant and is obtained from the mean value of the General Bathymetric Chart of the Oceans (GEBCO) over the region located below the sounded area. the NRLMSISE-00 empirical model [21], taken on the day and at the local time of the observation allows to model the atmospheric structure and compute atmospheric parameter such as the temperature, the density, the viscosity, the sound speed, etc...

Figure 1 shows the real part of the vertical (U(r)) and horizontal (V(r)) eigenfunctions in the atmosphere and the dispersion diagram of the tsunami normal modes computed for the Haida Gwaii event. As there is an important atmospheric resonance at 1.5 mHz, 2.0 mHz and 2.5 mHz, [22], the vertical and horizontal components (U(r) and V(r)) of tsunami normal mode eigenfunctions in the atmosphere will be large around these frequencies.

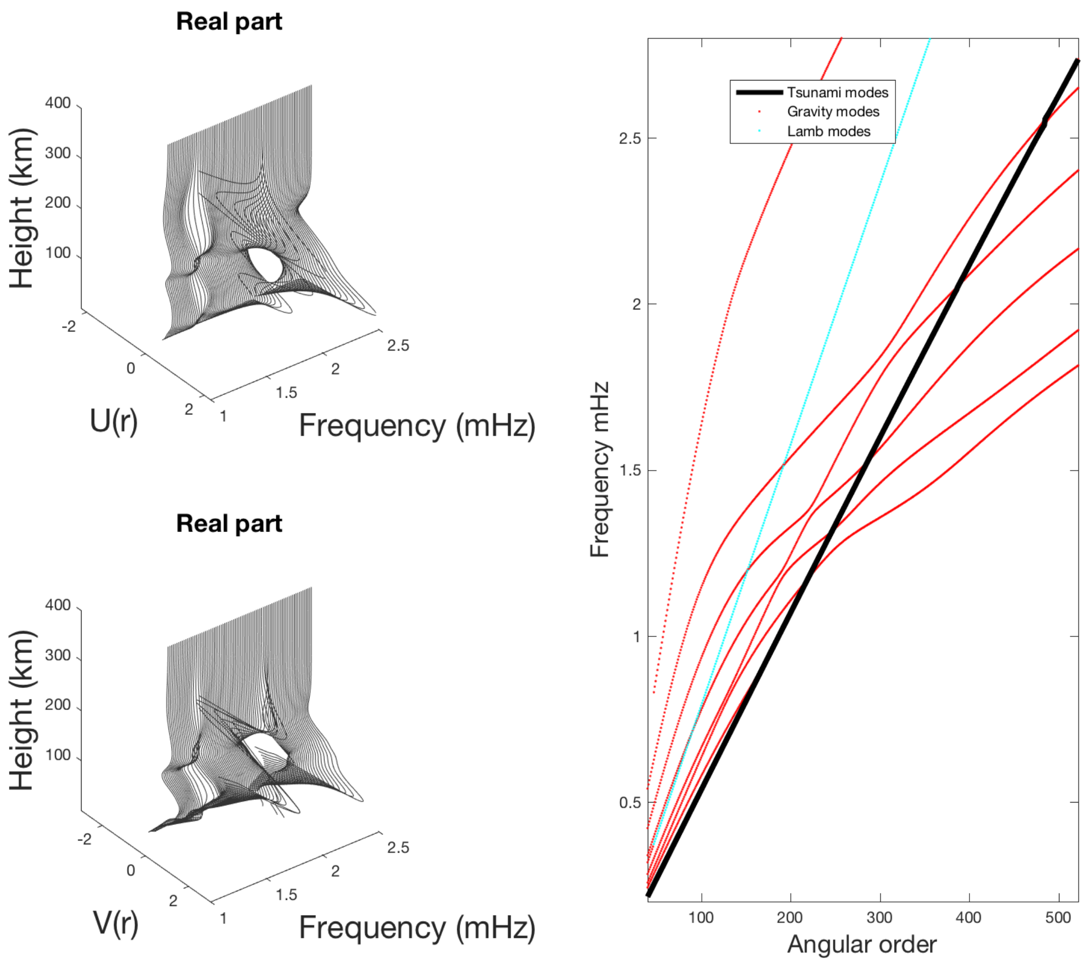


Figure 1: Normal mode characteristics. Lefthand panel: amplitude of the atmospheric part of vertical (Ur) and horizontal (Vr) normal mode eigenfunctions, scaled by the square root of density. Righthand panel: dispersion of normal modes for the full Earth system from 0.2 mHz to 2.6 mHz. Tsunami modes are plotted in black, atmospheric gravity modes in red, and Lamb modes in light blue.

# 4 Reminder on the general spherical harmonics representation

In this section, we recall the basis of the general spherical harmonics representation used by [20] to describe the Earth normal modes displacements.

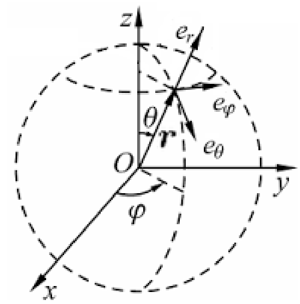


Figure 2: Spherical basis representation.

### 4.1 Einstein summation convention

Let's note $u^1$,$u^2$, $u^3$, ... $u^n$ the component of the vector $\mathbf{u}$ in any basis $\mathbf{e_1}$ ,$\mathbf{e_2}$, $\mathbf{e_3}$, ... $\mathbf{e_n}$.

$$\mathbf{u} = \sum_{i=1}^{N} u^i \mathbf{e}_i. \tag{1}$$

According to the Einstein convention, a repeated index is implicitly summed. Thus the equation (2) can be simplified as :

$$\mathbf{u} = u^i \mathbf{e}_i, \tag{2}$$

As the normal modes are expressed in a 3D basis, in all our study $N = 3$.

### 4.2 Spherical basis

Assuming Earth is a radially symmetric system, it is natural to use spherical coordinates $(r, \theta, \phi)$ to express the normal modes excitation. $r$ is the distance from the center of the Earth, the co-latitude is equal to $\theta = 90^o - \delta$ where $\delta$ is the latitude and $\phi$ is the longitude. A representation of the orthonormal spherical basis is shown in figure 2. Any vector $\mathbf{u}$ can be written as :

$$\mathbf{u} = u^i \mathbf{e_i} = u^r \mathbf{e}_r + u^\theta \mathbf{e}_\theta + u^\phi \mathbf{e}_\phi. \tag{3}$$

The scalar product is written:

$$u \cdot v = \delta_{ij} u^i v^j, \tag{4}$$

where $\delta_{ij}$ the Kronecker symbol :

$$\begin{cases} \delta_{ij} = & 1 \text{ if } i = j \\ \delta_{ij} = & 0 \text{ if } i \neq j. \end{cases} \tag{5}$$

## 4.3 Canonical basis.

The spherical geometry lead us to use spherical harmonics representation to resolve the normal modes equation :

$$Y_\ell^m(\theta,\varphi) = P_\ell^m(\cos\theta)\mathrm{e}^{+\,i\,m\,\varphi}, \tag{6}$$

with $\ell$ is the angular order, $m$ the azimuthal order where $-\ell \le m \le \ell$ and $P_\ell^m$ is the Legendre polynomial. However it is not possible to expand the $r$, $\theta$, $\phi$ components of vector or tensor field directly in terms of scalar spherical harmonics. Generally, a vector field $\mathbf{u}$ is expressed with the help of potential scalars (P,Q,R) as follows :

$$\mathbf{u}(\mathbf{r}) = P(\mathbf{r})\mathbf{e}_r + \boldsymbol{\nabla}_s Q(r) + \mathbf{e}_r \wedge \boldsymbol{\nabla}_s R(\mathbf{r}), \tag{7}$$

where the operator $\boldsymbol{\nabla}_s$ applied in a scalar function $F$ reads :

$$\boldsymbol{\nabla}_s F = \boldsymbol{\nabla} F - (\mathbf{e}_r \cdot \boldsymbol{\nabla} F)\,\mathbf{e}_r. \tag{8}$$

According to the group theory, the Earth system can be described by the Lie group SU(2) defined as :

$$SU(2) = \left\{ \begin{pmatrix} ia & -\overline{z} \\ z & -ia \end{pmatrix} :\ a \in \mathbf{R}, z \in \mathbf{C} \right\}. \tag{9}$$

The unit matrix of this group can be written as :

$$\begin{cases} \sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{cases} \tag{10}$$

$\sigma_1$, $\sigma_2$, and $\sigma_3$ are called the Pauli matrix widely used in quantum mechanics. Thus we can express the normal modes displacements through linear combinations, $u^-$ , $u^0$, $u^+$, expressed as:

$$\begin{cases} u^- = & \frac{1}{\sqrt{2}}(u_\theta + iu_\phi) \\ u^0 = & u_r \\ u^+ = & \frac{1}{\sqrt{2}}(-u_\theta + iu_\phi). \end{cases} \tag{11}$$

This linear combinations can be seen as the components of the vector $\mathbf{u}$ in a new basis called canonical basis. We can show that the new unit vector in this basis can be written as:

$$\begin{cases} \mathbf{e}_- = & \frac{1}{\sqrt{2}}(\mathbf{e}_\theta - i\mathbf{e}_\phi) \\ \mathbf{e_0} = & \mathbf{e_r} \\ \mathbf{e}_+ = & \frac{1}{\sqrt{2}}(-\mathbf{e}_\theta - i\mathbf{e}_\phi). \end{cases} \tag{12}$$

In canonical basis, the scalar product becomes:

$$\mathbf{u} \cdot \mathbf{v} = e_{\alpha\beta} u^{\alpha} v^{\beta}, \tag{13}$$

where the $e_{\alpha}\beta$ tensor components are:

$$\begin{cases} e_{00} &= & 1 \\ e_{+-} &= & e_{-+} = -1 \\ e_{\alpha\beta} &= & 0 \text{ if } \alpha+\beta \neq 0 \end{cases} \tag{14}$$

### 4.4 Generalized spherical harmonics

The components $u^{\alpha}$ in canonical basis of any vector can be expanded in terms of generalized spherical harmonics:

$$u^{\alpha}(r,\theta,\phi) = \sum_{n} \sum_{l=0}^{\infty} \sum_{m=-l}^{\ell} {}_{n}U_{\ell}^{\alpha m}(r) Y_{\ell}^{\alpha m}(\theta,\phi), \tag{15}$$

where $Y_{\ell}^{\alpha m}$ are the generalized spherical harmonics and ${}_{n}U_{\ell}^{\alpha m}(r)$ the radial function. Note that in the rest of this chapter, the index, $n$, $\ell$, and $m$ may be dropped when they are not necessary for the comprehension of the equations.

$$Y_{\ell}^{\alpha m}(\theta,\phi) = P_{\ell}^{\alpha m}(\mu) \mathrm{e}^{+\,i\,m\,\varphi}, \tag{16}$$

with $\mu = \cos\theta$. $P_{\ell}^{Nm}$ is the generalized Legendre Polynomial [20]. In a similar way any rank 2 tensor $m_{\alpha\beta}$ can be expanded in generalized spherical harmonics:

$$m^{\alpha\beta}(r,\theta,\phi) = \sum_{n} \sum_{l=0}^{\infty} \sum_{m=-l}^{\ell} {}_{n}M_{\ell}^{(\alpha+\beta)m}(r) Y_{\ell}^{(\alpha+\beta)m}(\theta,\phi). \tag{17}$$

This relation can be extended to an N order tensor. Let's note $D^{\gamma} m^{\alpha\beta}$ the derivative of $m^{\alpha\beta}$ relative to $\gamma$ index. It can be written in the following form :

$$D^{\gamma} m^{\alpha\beta} = D^{\gamma} M^{\alpha\beta}(r) Y_{\ell}^{(\alpha+\beta+\gamma)m}(\theta,\phi). \tag{18}$$

From [20], we have:

$$\begin{cases} D^{-} M^{\alpha\beta}(r) = & \frac{1}{r}\left[\Omega_{N}^{\ell} M^{\alpha\beta}(r) - X^{-}\right] \\ D^{0} M^{\alpha\beta}(r) = & \frac{dM^{\alpha\beta}(r)}{dr} \\ D^{-} M^{\alpha\beta}(r) = & \frac{1}{r}\left[\Omega_{N}^{\ell} M^{\alpha\beta}(r) - X^{+}\right], \end{cases} \tag{19}$$

where $N = \alpha + \beta$ and $\Omega_{N}^{\ell} = \left[(l+N)(l-N+1)/2\right]^{\frac{1}{2}}$ .

$X^-$ is the sum of terms obtained from $M^{\alpha\beta}$ by changing superscripts + into 0 and 0 into - one at a time and $X^+$ is the sum of terms obtained from $M^{\alpha\beta}$ by changing superscripts 0 into + and -into 0 one at a time.

As example we compute $D^- M^{+0}$:

$$D^- M^{+0}(r) = \frac{1}{r}\left(\Omega_1^\ell M^{+0} - M^{00} - M^{+-}\right). \tag{20}$$

with $\Omega_1^\ell = \sqrt{l(l+1)/2}$. The canonical derivative operators are:

1. $e_{\alpha\beta} D^{\alpha\beta}\Phi$ the laplacian for a scalar field.
2. $D^\beta u^\alpha$ the gradient of vector field.
3. $e_{\alpha\beta} D^\beta u^\alpha$ the divergence of a vector.
4. $e_{\beta\gamma} D^\gamma \tau^{\alpha\beta}$ the divergence of a second rank tensor.

# 5 Solutions for the normal modes

In this section, we recall the main results obtained by [20] about the Earth normal modes solutions.

## 5.1 Strain and Stress

Let us note the stress $\tau^{\alpha\beta}$ and the strain $\epsilon^{\alpha\beta}$. The strain is defined as:

$$\epsilon^{\alpha\beta} = \frac{1}{2}\left(\partial^\beta u^\alpha + \partial^\alpha u^\beta\right). \tag{21}$$

We expanded these 2 tensors in generalized spherical harmonics :

$$\begin{cases} \epsilon^{\alpha\beta}(r,\theta,\phi) = E^{\alpha\beta}(r) Y_\ell^{(\alpha+\beta)m}(\theta,\phi) \\ \tau^{\alpha\beta}(r,\theta,\phi) = T^{\alpha\beta}(r) Y_\ell^{(\alpha+\beta)m}(\theta,\phi). \end{cases} \tag{22}$$

## 5.2 Computation of the derivative

The spheroidal modes (which include tsunami ones) have the following properties due to symetry considerations:

$$\begin{cases} U^+ = & U^- \\ T^{0+} = & T^{0-} \\ T^{++} = & T^{--}. \end{cases} \tag{23}$$

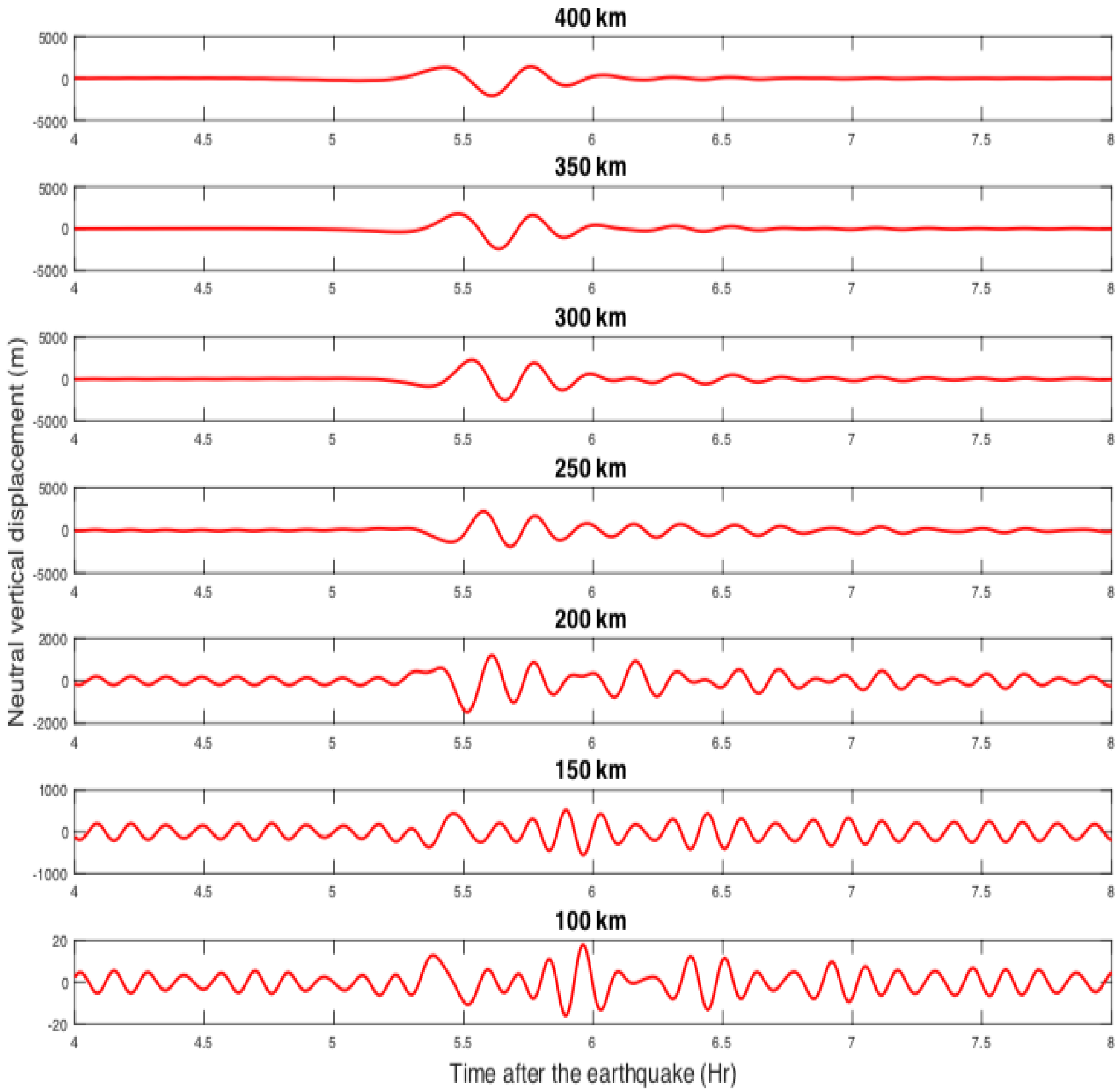


Figure 3: Vertical neutral displacement induced by the 2012 Haida Gwaii tsunami in the atmosphere between 100 km and 400 km in Hawaii (DART 51407) for the Haida Gwaii tsunami.

From equation (19), we compute the components of the derivative tensor $D^{\beta}u^{\alpha}$ :

$$
\left\{
\begin{array}{rc}
D^0U^0(r,\theta,\phi) = & \frac{dU^0}{dr}(r) \\
D^+U^-(r,\theta,\phi) = & \frac{1}{r}\left(\sqrt{\frac{l(l+1)}{2}}U^-(r) - U^0(r)\right) \\
D^0U^+(r,\theta,\phi) = & \frac{dU^+}{dr}(r) \\
D^+U^0(r,\theta,\phi) = & \frac{1}{r}\left(\sqrt{\frac{l(l+1)}{2}}U^0(r) - U^+(r)\right) \\
D^0U^-(r,\theta,\phi) = & \frac{dU^+}{dr}(r) \\
D^-U^0(r,\theta,\phi) = & \frac{1}{r}\left(\sqrt{\frac{l(l+1)}{2}}U^0(r) - U^+(r)\right) \\
D^+U^+(r,\theta,\phi) = & \frac{1}{r}\left(\sqrt{\frac{(l-1)(l+2)}{2}}U^+(r)\right) \\
D^-U^-(r,\theta,\phi) = & \frac{1}{r}\left(\sqrt{\frac{(l-1)(l+2)}{2}}U^+(r)\right),
\end{array}
\right.
\tag{24}
$$

with $D^\alpha u^\beta = D^\alpha U^\beta Y_\ell^{(\alpha+\beta)m}$.

In this section, we recall the synthetics seismograms computation method developed by [20]. The double couple (or point dislocation) is the simplest model which allows to represent the source mechanism of an earthquake. In this section, we consider the case of an ideal point source. This is done by applying a stress at the epicenter $\mathbf{r} = \mathbf{r_0}$ at time $t = 0$. In this model, the stress tensor reads :

$$
\tau^{\alpha\beta} = M^{\alpha\beta}\delta(\mathbf{r} - \mathbf{r_0})H(t), \tag{25}
$$

where $H(t)$ is the Heaviside function and $\mathbf{M}$ the moment tensor. The corresponding applied volumic force is :

$$
f = \boldsymbol{\nabla} \cdot \boldsymbol{\tau}. \tag{26}
$$

The excitation ${}_n\boldsymbol{\psi}_\ell$ of the normal mode $(n,\ l,\ m)$ is then :

$$
{}_n\psi_\ell^m = -\int \boldsymbol{M} : {}_n\boldsymbol{\epsilon}_\ell^{m*}\delta(\mathbf{r} - \mathbf{r_0})dV. \tag{27}
$$

In canonical components with the computation of the double product, we have:

$$
{}_n\psi_\ell^m = M^{\alpha\beta} : {}_n\epsilon_\ell^{\alpha\beta*}(\mathbf{r_0})\delta_{\alpha\beta}\delta_{\gamma\delta}. \tag{28}
$$

The canonical components of $\mathbf{M}$ are :

$$
\left\{
\begin{array}{rc}
M^{00} = & M_{rr} \\
M^{+-} = & \frac{1}{2}\left(M_{\theta\theta} + M_{\phi\phi}\right) \\
M^{0\pm} = & \frac{1}{\sqrt{2}}\left(M_{r\theta} + iM_{r\phi}\right) \\
M^{\pm\pm} = & \frac{1}{2}\left(M_{\theta\theta} - M_{\phi\phi}\right) \mp iM_{\theta\phi}.
\end{array}
\right.
\tag{29}
$$

Using equation (22 ) in spherical harmonics we have :

$$
\left\{
\begin{array}{lc}
\mathbf{M} : {}_n\boldsymbol{\epsilon}_\ell{}^* = & M^{00}{}_n\epsilon_\ell{}^{00*} + 2M^{+-}{}_n\epsilon_\ell{}^{+-*} + \\
& 2M^{+-}{}_n\epsilon_\ell{}^{+-*} + 2M^{0+}{}_n\epsilon_\ell{}^{0+*} + 2M^{0-}{}_n\epsilon_\ell{}^{0-*} + \\
& M^{++}{}_n\epsilon_\ell{}^{++*} + M^{--}{}_n\epsilon_\ell{}^{--*}.
\end{array}
\right.
\tag{30}
$$

We apply a rotation such as the source is at the pole $\mathbf{r}_0 = (r_{Earth}, \theta_{r_0} = 0, \phi_{r_0} = 0)$. The equation (30) becomes :

$$
\left\{
\begin{array}{lc}
\mathbf{M} : {}_n\boldsymbol{\epsilon}_\ell{}^* = & \left(M^{00}{}_nE_\ell{}^{00*} + 2M^{+-}{}_nE_\ell{}^{+-*}\right) P_\ell^{0m}(1) + \\
& 2\left(M^{0+}{}_nE_\ell{}^{0+*}\right) P_\ell^{1m}(1) + \\
& 2\left(M^{0-}{}_nE_\ell{}^{0-*}\right) P_\ell^{-1m}(1) + \\
& \left(M^{++}{}_nE_\ell{}^{++*}\right) P_\ell^{2m}(1) + \\
& \left(M^{--}{}_nE_\ell{}^{--*}\right) P_\ell^{-2m}(1).
\end{array}
\right.
\tag{31}
$$

We have the following relationship on generalized Legendre polynomial:

$$
\left\{
\begin{array}{l}
P_\ell^{Nm} = 1 \text{ if } N = m \\
P_\ell^{Nm} = 0 \text{ if } N \neq m.
\end{array}
\right.
\tag{32}
$$

Thus equation 31 becomes :

$$
\left\{
\begin{array}{rc}
{}_n\psi_\ell^0 = & -\left(M^{00}{}_nE_\ell^{00} + 2M^{+-}{}_nE_\ell^{+-}\right)(r_0) \\
{}_n\psi_\ell^{\pm} = & -2M^{0\pm}{}_nE_\ell^{0\pm}(r_0) \\
{}_n\psi_\ell^{\pm 2} = & -M^{\pm\pm}{}_nE_\ell^{\pm\pm}(r_0).
\end{array}
\right.
\tag{33}
$$

This means that only the modes for $m = \pm 2, \pm 1, 0$ will contribute in the computation of the synthetic seismograms.

In addition, using(21),(22),(24) and (23) we find the following relation:

$$
\left\{
\begin{array}{rcl}
E^{0+} & = & E^{0-} \\
E^{++} & = & E^{--}.
\end{array}
\right.
\tag{34}
$$

Finally, from (33), (34) and (29) we can deduce :

$$
{}_n\psi_\ell^m = (-1)^m {}_n\psi_\ell^{m*}.
\tag{35}
$$

Then the displacement can be written:

$$u^\alpha(r_s,\theta_s,\phi_s,t) = \sum_n \sum_\ell \frac{1-e^{i_n\omega_\ell t}}{{}_n\omega_\ell^2} \times \\ {}_nU_\ell^\alpha \left( \sum_{m=-2}^{2} {}_n\psi_\ell^m P_\ell^{\alpha m}(\cos\theta_s) e^{im\phi_s} \right). \tag{36}$$

In figure 3, we show the displacement of the neutral due to the propagation of the gravity wave induced by the tsunami at Hawaii (location of the DART buoy 51407) between 100 km and 400 km. In addition, as the air density decays exponentially, the amplitude of the neutral displacement increases exponentially (energy conservation principle).

As expected, the high atmosphere filter a high band filter. This can be explained by the fact, that the high frequency tsunami normal modes are already evanescent at high altitude and thus do not propagate.

# 6 Computation of the Perturbed TEC normal modes

In this last section, we describe a new computation method of TEC perturbation using the general spherical harmonics representation [20].

The ion continuity equations reads :

$$\frac{\partial n_e}{\partial t} + \boldsymbol{\nabla} \cdot (n_e \mathbf{v_i}) = 0, \tag{37}$$

where $\mathbf{v}_i$ is the ion velocity and $n_e$ the electron density.

The ion velocity $\mathbf{v_i}$ is:

$$\mathbf{v_i} = \frac{1}{1+\kappa^2} \left[ \kappa^2 \mathbf{v} + \kappa \mathbf{v} \times \mathbf{1_b} + (\mathbf{u} \cdot \mathbf{1_b}) \mathbf{1_b} \right], \tag{38}$$

with $\mathbf{1_b}$ the geomagnetic field unit vector, $\kappa$ the ratio of the neutral ion collision frequency with the gyrofrequency $\gamma_i = q_i B/m_i$, where $B$ is the magnetic field intensity, $q_i$ the ion charge and $m_i$ its mass.

Due to the fact, the ion-neutral collision frequency decreases at high altitude, $\kappa$ can be neglected in the F region and the equation becomes:

$$\mathbf{v_i} = (\mathbf{v} \cdot \mathbf{1_b})\, \mathbf{1_b} \tag{39}$$

We assume the electron density fluctuations are small : $n_e = n_e^0 + \delta n_e$, and after integration the continuity equation becomes :

$$\delta n_e = -\boldsymbol{\nabla} \cdot \left( n_e^0 \mathbf{u_i} \right), \tag{40}$$

with $\mathbf{u_i}$ the displacement of the ions. The ions velocity in the case of the tsunami of Haida Gwaii is Hawaii is shown in figure **??**.

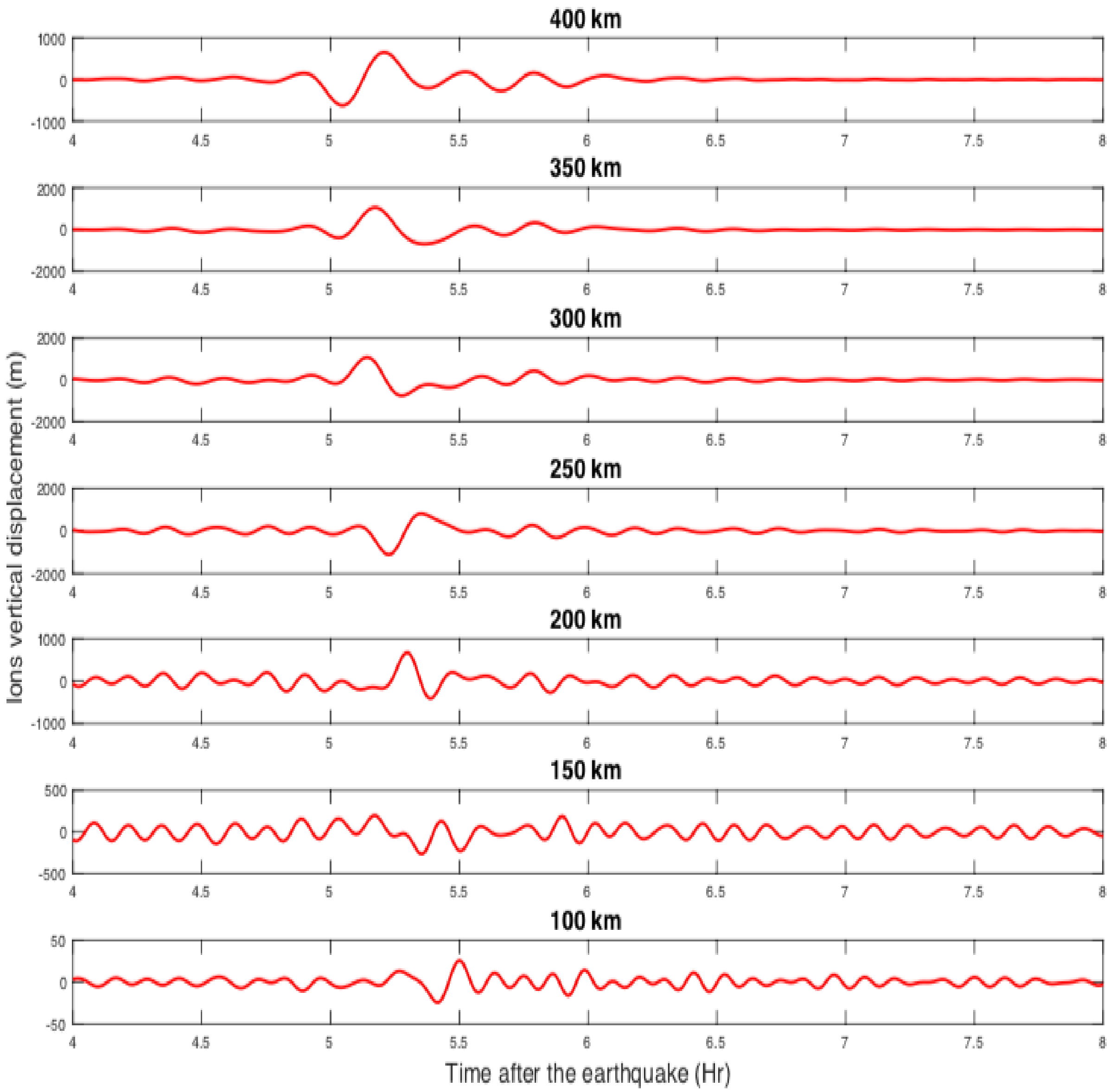


Figure 4: Ions neutral displacement induced by the 2012 Haida Gwaii tsunami in between 100 km and 400 km altitude in Hawaii (DART 51407) for the Haida Gwaii tsunami.

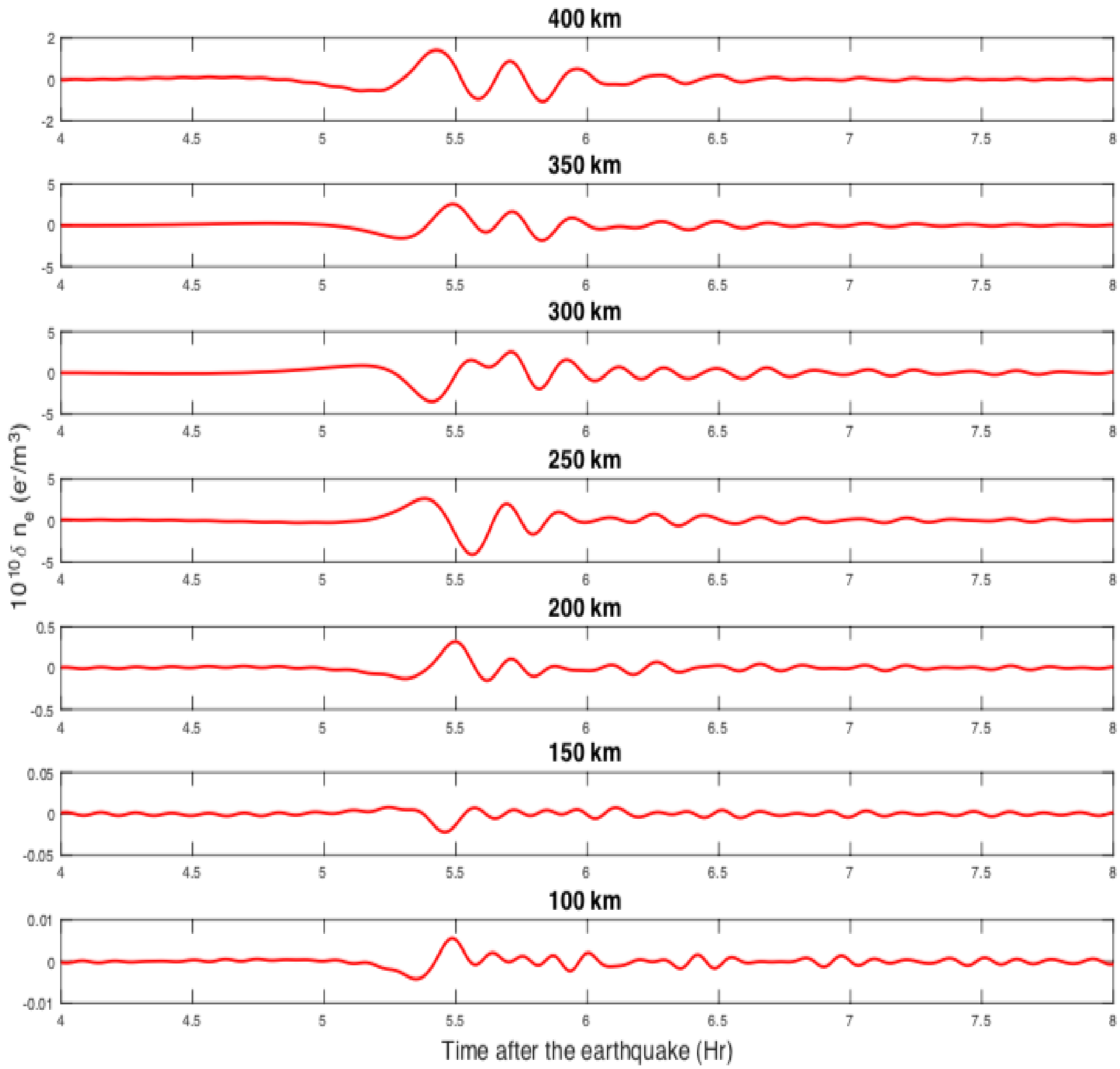


Figure 5: Electron density perturbation induced by the 2012 Haida Gwaii tsunami in between 100 km and 400 km altitude in Hawaii (DART 51407).

Then we can rewrite (40) in the index notation.

$$\delta n_e = -e_{\alpha\beta} e_{\delta\gamma} D^{\beta} \left( n_e^0 b^{\alpha} b^{\gamma} u^{\delta} \right). \tag{41}$$

We decompose this derivative in 4 terms:

$$\begin{aligned}\delta n_e = &- e_{\alpha\beta}e_{\delta\gamma}\times \\ & b^\gamma u^\delta b^\alpha D^\beta n_e^0 + n_e^0\left(b^\alpha u^\delta D^\beta b^\gamma + b^\gamma u^\delta D^\beta b^\alpha + b^\gamma b^\alpha D^\beta u^\delta\right).\end{aligned} \tag{42}$$

We can compute the first 3 terms in spherical coordinates:

$$\begin{cases} e_{\alpha\beta}e_{\delta\gamma}b^\gamma u^\delta b^\alpha D^\beta n_e^0 = & (\mathbf{b}\cdot\mathbf{u})\,\mathbf{b}\cdot\boldsymbol{\nabla} n_e^0 \\ e_{\alpha\beta}e_{\delta\gamma}n_e^0 u^\delta b^\alpha D^\beta b^\gamma = & n_e^0\left[(\mathbf{b}\cdot\boldsymbol{\nabla})\,\mathbf{b}\right]\cdot\mathbf{u} \\ e_{\alpha\beta}e_{\delta\gamma}n_e^0 u^\delta b^\gamma D^\beta b^\alpha = & n_e^0\,(\mathbf{b}\cdot u)\,\boldsymbol{\nabla}\cdot\mathbf{b}. \end{cases} \tag{43}$$

The normal mode displacement $u_\ell^{m\alpha}$ for a given $m$ is :

$$u_\ell^{m\alpha} = {}_n\psi_\ell^m U^\alpha Y_\ell^{\alpha m}. \tag{44}$$

Thus, the last term of equation 42 can be computed using equation (24). We have:

$$e_{\alpha\beta}e_{\delta\gamma}n_e^0 b^\gamma b^\alpha D^\beta u^\delta = n_e^0(A+B+C+D+E). \tag{45}$$

We have taken the following notations:

$$\begin{cases} A_{n,l,m} = & b^0 b^0\,{}_n\psi_\ell^m D^0 U^0 Y_\ell^{0m} \\ B_{n,l,m} = & b^+ b^-\,{}_n\psi_\ell^m\,(D^- U^+ + D^+ U^-)\,Y_\ell^{0m} \\ C_{n,l,m} = & -b^0\,{}_n\psi_\ell^m\left(b^+ D^0 U^- Y_\ell^{-1m} + b^- D^0 U^+ Y_\ell^{1m}\right) \\ D_{n,l,m} = & -b^0\,{}_n\psi_\ell^m\left(b^+ D^- U^0 Y_\ell^{-1m} + b^- D^+ U^0 Y_\ell^{1m}\right) \\ E_{n,l,m} = & {}_n\psi_\ell^m\left(b^+ b^+ D^- U^- Y_\ell^{-2m} + b^+ b^+ D^+ U^0 Y_\ell^{2m}\right). \end{cases} \tag{46}$$

We have:

$$A_{n,l,m} = b^0 b^0\,{}_n\psi_\ell^m \partial^0 U^0 Y_\ell^{0m}. \tag{47}$$

Thus

$$B_{n,l,m} = b^+ b^-\,{}_n\psi_\ell^m\left(D^- U^+ + D^+ U^-\right)Y_\ell^{0m}. \tag{48}$$

From (24) we conclude for $B$:

$$B_{n,l,m} = 2b^+ b^-\,{}_n\psi_\ell^m \partial^- U^+ Y_\ell^{0m}. \tag{49}$$

For the other terms we sum the $m$ and $-m$ contribution in order to express everything in function of real spherical harmonics. Using equation (35), the normal modes displacement contribution summed for $m$ and $-m$ is :

$$u^{\alpha}_{m-m} = \partial^0 U^+ \left( {}_n\psi^m_\ell P^{\alpha m}_\ell e^{im\phi_s} + (-1)^m {}_n\psi^{m*}_\ell P^{\alpha-m}_\ell e^{-im\phi_s} \right). \tag{50}$$

Let us write $C_{n,l,m-m} = C_{n,l,m} + C_{n,l,-m}$. Thus we obtained :

$$\begin{aligned} C_{n,l,m-m} = &- D^0 U^+ b^0 {}_n\psi^m_\ell (b^- P^{1m}_\ell e^{im\phi_s} + b^+ P^{-1m}_\ell e^{im\phi_s}) - \\ & D^0 U^+ b^0 (-1)^m {}_n.\psi^{m*}_\ell (b^- P^{1-m}_\ell e^{-im\phi_s} + b^+ P^{-1-m}_\ell e^{-im\phi_s}) \end{aligned} \tag{51}$$

In addition we have the following relation on generalized legendre polynomial :

$$P^{\alpha m}_\ell = (-1)^{m+\alpha} P^{-\alpha-m}_\ell. \tag{52}$$

In using (52) in (51) we obtain:

$$\begin{aligned} C_{n,\ell,m-m} = &-D^0 U^+ b^0 P^{1m}_\ell \left( {}_n\psi^m_\ell b^- e^{im\phi_s} - {}_n\psi^{m*}_\ell b^+ e^{-im\phi_s} \right) - \\ & D^0 U^+ b^0 P^{1-m}_\ell \left( {}_n\psi^{-m}_\ell b^- e^{-im\phi_s} - {}_n\psi^{-m*}_\ell b^+ e^{im\phi_s} \right). \end{aligned} \tag{53}$$

From (11) we have $b^+ = (-b^-)^*$.
We can conclude:

$$C_{n,l,m-m} = -2D^0 U^+ b^0 \left( Re({}_n\psi^m_\ell b^- Y^{1m}_\ell) + Re({}_n\psi^{-m}_\ell b^- Y^{1-m}_\ell) \right). \tag{54}$$

and so:

$$C_{n,l,m} = -2D^0 U^+ b^0 Re({}_n\psi^m_\ell b^- Y^{1m}_\ell). \tag{55}$$

With the same type of calculations we compute $D^m$ and $E_m$:

$$\left\{ \begin{aligned} D_{n,l,m} &= -2D^+ U^0 b^0 Re({}_n\psi^m_\ell b^- Y^{1m}_\ell) \\ E_{n,l,m} &= 2D^+ U^+ Re({}_n\psi^m_\ell b^- b^- Y^{2m}_\ell). \end{aligned} \right. \tag{56}$$

The electron perturbation density is shown on figure 5.

The perturbed TEC normal modes is then obtained by integration of the perturbed electronic density the long of the GPS satellite- ground station line of sight.

# 7 Comparison with the data observations

We now compare our TEC modeling with the observation derived from 30 s data of the SOPAC (http://sopac.ucsd.edu/) Hawaii GPS array for one station (kosm) and 4 satellites. We compare the data with both point source model and extended source model. The results are shown figure 6.

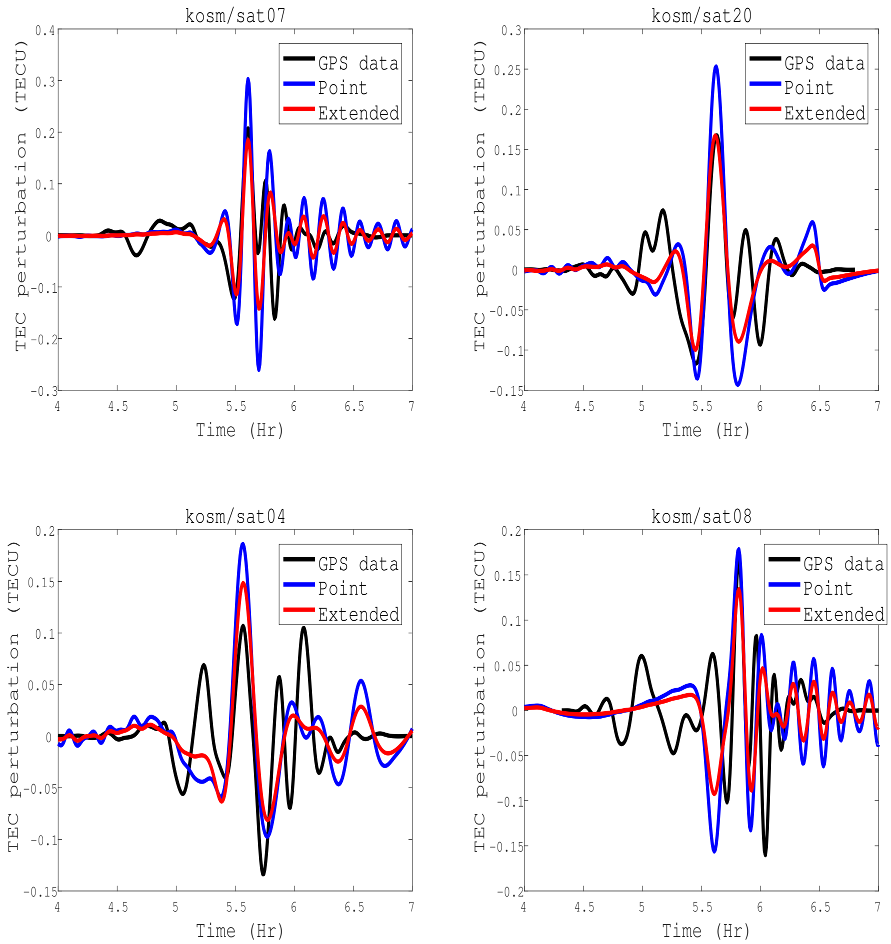


Figure 6: 2012 Haida Gwaii tsunami situation map and forward modeling results. a) Map centered on the location the 2012 Haida Gwaii tsunami. b) Map centered on Hawaii. The white line marks the trace of the kosm station and satellite 07 from 4h to 7h after the earthquake. c) Perturbed TEC for station kosm and satellite 07. d) Tsunami amplitude for the DART buoy 51407. We filtered data (black curve) and synthetics (blue and red curves for the point source and extended source case, respectively) between 0.2 mHz and 2.6 mHz. Both point source and extended source modeling are shown. A time shift of -9.5 mn is applied to the model.

The normal modes model uses atmospheric parameters computed at the time and location of the tsunami arrival and with the ocean depth at Hawaii (DART buoy 51407). We used 2 types of sources : a point source located at the epicenter and an extended source corresponding to several point sources distributed along the fault plane. Source parameters used in the simulations for the point source were those of the Centroid Moment Tensor (CMT) project [6] while those for the extended sources were the finite fault source models from the U.S. Geophysical Survey (http://earthquake.usgs.gov/) for the three events [9]. Due to the fact that we neglected the variation of the ocean depth, our modeling is not able to predict the exact arrival time of the perturbation of TEC induced by tsunami. Thus, we have to shift our model by 9.5 mn. We are able to reproduce the TEC perturbation main peak for the satellites. As expected, the amplitude obtained for the point source model is too high as it supposed that all energy is emitted at the epicenter.

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