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### A stability analysis of a conduit flow model for lava dome eruptions

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#### Abstract

Periodic variations in magma discharge rate and ground deformation have been commonly observed during lava dome eruptions. We performed a stability analysis of a conduit flow model by Barmin et al. [Barmin, A., Melnik, O., Sparks, R.S.J., 2002. Periodic behavior in lava dome eruptions. Earth and Planetary Science Letters 199 (1-2), 173–184], in which the periodic variations in magma flow rate and chamber pressure are reproduced as a result of the temporal and spatial changes of the magma viscosity controlled by the kinetics of crystallization. The model is reduced to a dynamical system where the time derivatives of the magma flow rate (dQ/dt) and the chamber pressure (dP/dt) are functions of Q and P evaluated at a shifted time  $t-t_*$ . Here, the time delay  $t_*$  represents the time for the viscosity of fluid particle to increase in a conduit. The dynamical system with time delay is approximated by a simple two-dimensional dynamical system of Q and P where  $t_*$  is given as a parameter. The results of our linear stability analyses for these dynamical systems indicate that the transition from steady to periodic flow depends on nonlinearities in the steady state relation between Q and P. The steady state relation shows a sigmoidal curve in Q-P phase plane; its slope has negative values at intermediate flow rates. The steady state solutions become unstable, and hence P and Q oscillate periodically, when the negative slope of the steady state relation ( $[dP/dQ]_S$ ) exceeds a critical value; that is  $[dP/dQ]_{S} < -t_*\gamma/(2V_{ch})$ , where  $V_{ch}$  is the chamber volume and  $\gamma$  is an elastic constant which is related to the rigidity of chamber wall. We also found that the period and the pattern of oscillation of the conduit flow primarily depend on a quantity defined by  $LV_{ch}/r^4$ , where L is the conduit length and r is the conduit radius. © 2008 Elsevier B.V. All rights reserved.

Keywords: conduit flow; stability analysis; dynamical system; time delay

#### 1. Introduction

During lava dome eruptions, magma effusion rate commonly changes with periods of a few years (e.g., the 1922–2002 eruption of Santiaguito (Rose, 1973; Harris et al., 2003), the 1980–1986 eruption of Mount St. Helens (Swanson and Holcomb, 1990) and the 1991–1995 eruption of Unzen (Nakada et al., 1999)). Such periodic variations have been explained by coupling effects of conduit flow with variable magma viscosity and pressure in magma chamber with elastic wall. It has been suggested that the magma flow rate and the chamber pressure tend to oscillate when effective viscosity of magma decreases as magma flow rate increases (e.g., Whitehead and Helfrich, 1991; Ida, 1996; Wylie et al., 1999; Maeda, 2000; Melnik and Sparks, 2005).

The relationship between effective viscosity of magma (gascrystal-liquid mixture) and conduit flow is complex, because the effective viscosity is governed by many physical factors, and because it changes spatially and temporally within conduits. The effective viscosity of magma is a function of temperature (Hess and Dingwell, 1996), dissolved volatile content of the magma (Richet et al., 1996), distribution and shape of bubbles (Pal, 2003), degree of crystallization (e.g., Lejeune and Richet, 1995; Costa, 2005; Costa et al., 2007a,b) and so on. The temperature distribution and distributions of bubbles, volatiles and crystals within the conduit evolve depending on magma residence time in the conduit, and hence magma flow rate (e.g., Melnik and Sparks, 1999, 2002). Furthermore, the change in conduit radius also modifies the viscous resistance of conduit flow (Slezin, 2003). Some of these coupling effects between the magma flow rate, the chamber pressure and the effective viscosity have been investigated on the basis of simplified models where the temporal change in the spatially averaged

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viscosity is given as a certain function of the magma flow rate and the chamber pressure (Whitehead and Helfrich, 1991; Ida, 1996; Wylie et al., 1999; Maeda, 2000). In addition to these simplified models, Melnik and Sparks (2005), Costa et al. (2007a) and Costa et al. (2007b) have recently proposed fluid dynamical models for one-dimensional conduit flow where both the temporal and spatial changes of the rheological properties of the magma are described by a series of partial differential equations. These models have reproduced the periodic behavior of the magma flow rate during lava dome eruptions; however, it is rather difficult to understand the essential physics behind such sophisticated models.

In this study, we investigate mathematical features of a model proposed by Barmin et al. (2002). The coupling effects of the conduit flow and the chamber pressure as well as the temporal and spatial changes of the effective viscosity are taken into account in the simplest way in this model. In this sense, we regard this model as a minimal model that captures the essential physics of the previous models for periodic behavior in lava dome eruptions due to the viscosity change. Although Barmin et al. (2002) have numerically shown that the minimal model successfully reproduces the periodic behavior of magma flow rate and chamber pressure during lava dome eruptions, the mathematical features of the model have not been fully investigated. Here, we determine the condition for the occurrence of periodic behavior in Barmin's model on the basis of the linear stability theory.

#### 2. Model

Barmin et al. (2002) proposed a model of magma plumbing system where a magma chamber with elastic wall is being fed at a constant flux from greater depth and the magma ascends along a cylindrical conduit from the magma chamber to the surface (Fig. 1). It is assumed that viscosity of the magma primarily



Fig. 1. Schematic view of the volcanic system proposed by Barmin et al. (2002). Magma is supplied at a constant flux  $Q_{in}$  in a chamber (with an elastic constant  $\gamma$  and a volume  $V_{ch}$ ). The magma in a conduit ascends at a flux Q from the chamber. The viscosity of each fluid particle increases from  $\mu_1$  to  $\mu_2$  after the fluid particle ascends for a constant time  $t_*$  from the magma chamber.

depends on volume concentration of crystals with constant growth rate. As a result, the viscosity of each fluid particle increases from  $\mu_1$  to  $\mu_2$  after the fluid particle ascends for a constant time  $t_*$  from the magma chamber. As the magma flux increases, the position of the viscosity increase in the conduit ( $x_*$ in Fig. 1) becomes shallower, so that its average viscosity throughout the conduit decreases, which leads to an increase in the magma flux. On the other hand, as the magma flux increases, the pressure of the magma chamber with elastic wall decreases, which in turn suppresses the increase in the magma flux. These two mechanisms produce periodic variations in the magma flux and the chamber pressure.

The equation for the conduit flow is

$$\frac{\partial p}{\partial x} = -\rho g - \frac{8\mu Q}{\pi r^4} \begin{cases} p = 0 & \text{at} \quad x = L, \\ p = p_{\text{ch}} & \text{at} \quad x = 0, \end{cases}$$
(1)

where *p* is the pressure in the conduit,  $p_{ch}$  is the chamber pressure, *g* is the gravitational acceleration,  $\mu$  is the magma viscosity ( $\mu = \mu_1$  for  $0 < x < x_*$  and  $\mu = \mu_2$  for  $x_* < x < L$ ; see Fig. 1), *Q* is the flow rate through the conduit,  $\rho$  is the density of the magma, *r* is the radius of the conduit and *L* is the length of the conduit. This equation states the conservation of momentum in which the inertial term is negligibly small and conduit resistance is taken in Poiseuille form. The change of  $p_{ch}$  causing elastic deformation of the magma chamber wall is produced by difference between inflow  $Q_{in}$  and outflow *Q*. Using a chamber overpressure  $P(\equiv p_{ch} - \rho gL)$ , these relationships are given by

$$P = \frac{8\mu L}{\pi r^4} Q,\tag{2}$$

$$\overline{\mu} = \begin{cases} \mu_1 & x_* > L, \\ \frac{\mu_1 x_* + \mu_2 (L - x_*)}{L} & x_* < L, \end{cases}$$
(3)

$$\frac{dP}{dt} = \frac{\gamma}{V_{\rm ch}} (Q_{\rm in} - Q), \tag{4}$$

where  $\gamma$  is an elastic constant which is related to the rigidity of the wall rock of the magma chamber (see Melnik and Sparks (2005) and Costa et al. (2007b) for its explicit form),  $V_{\rm ch}$  is the chamber volume and  $\overline{\mu}$  is the average viscosity throughout the conduit.

In Barmin et al. (2002), the viscosity change is described by a series of equations related to the kinetics of crystal growth and the dependence of viscosity on crystal contents. However, mathematical features of Barmin's model are independent of details of the kinetics of crystal growth or those of the dependence of viscosity on crystal contents; they are determined only by the fact that the viscosity of each fluid particle increases after the fluid particle ascends for a constant time  $t_*$  from the magma chamber. Using  $t_*$ , the position of the viscosity increase  $x_*$  is given by

$$x_*(t) = \int_{t-t_*}^t \frac{Q(s)}{\pi r^2} ds.$$
 (5)

Eqs. (2)–(5) are the complete set of the equations for Barmin's model.

We normalize these equations by substituting  $Q = \overline{Q}Q'$ ,  $P = \overline{P}P'$ ,  $\overline{\mu} = \mu_1 \overline{\mu}'$ ,  $t = \overline{T}t'$  and x = Lx' where  $\overline{P}$ ,  $\overline{Q}$  and  $\overline{T}$  are

$$\begin{cases} \overline{P} = \frac{\pi r^2 L}{V_{\rm ch}} \gamma, \\ \overline{Q} = \frac{\overline{P} \pi r^4}{8\mu_1 L} = \frac{\pi^2 r^6 \gamma}{8\mu_1 V_{\rm ch}}, \\ \overline{T} = \frac{\pi r^2 L}{\overline{Q}} = \frac{8\mu_1 L V_{\rm ch}}{\pi r^4 \gamma}. \end{cases}$$
(6)

Similarly, the parameters  $Q_{in}$  and  $t_*$  are normalized by these characteristic scales as  $Q_{in} = \overline{Q}Q'_{in}$  and  $t_* = \overline{T}t'$ , respectively. The physical meaning of the characteristic scales in Eq. (6) will be discussed in a later section. Using these dimensionless parameters, we get

$$P'(t') = \overline{\mu}'(t')Q'(t').$$
(7)

$$\overline{\mu}'(t') = \begin{cases} 1 & x'_* > 1, \\ x'_* + \hat{\mu}(1 - x'_*) & x'_* < 1, \end{cases}$$
(8)

$$x'_{*}(t') = \int_{t'-t'_{*}}^{t'} \mathcal{Q}'(s) ds, \qquad (9)$$

$$\frac{dP'(t')}{dt'} = Q'_{\rm in} - Q'(t'), \tag{10}$$

where  $\hat{\mu} = \mu_2 / \mu_1$ .

Differentiating Eqs. (7)–(9) with respect to time t' for  $x'_* < 1$  yields

$$\frac{d\mathcal{Q}'(t')}{dt'} = \frac{\mathcal{Q}'(t')}{P'(t')} \left[ \mathcal{Q}'_{\rm in} - \mathcal{Q}'(t') + (\hat{\mu} - 1) \{ \mathcal{Q}'(t') - \mathcal{Q}'(t' - t'_{*}) \} \mathcal{Q}'(t') \right].$$
(11)

The time evolution of this system is described by the set of Eqs. (10) and (11). This system is a dynamical system with time delay, because dQ'/dt' depends on  $Q'(t'-t'_*)$ . It should also be noted that this system is essentially a dynamical system with one variable Q', because  $\overline{\mu}'$  (and hence  $P' = \overline{\mu}'Q'$ ) is given as a function of Q' (see Eqs. (7)–(9)). For  $x'_* > 1$ , since P'(t') = Q'(t') from Eqs. (7) and (8), Eq. (11) is replaced by

$$\frac{dQ'(t')}{dt'} = Q'_{\rm in} - Q'(t').$$
(12)

Our formulation and the original formulation by Barmin et al. (2002) are mathematically equivalent, although they have different forms; our formulation includes three dimensionless parameters ( $\hat{\mu}$ ,  $Q_{in'}$  and  $t'_*$ ), whereas the original one includes four dimensionless parameters. We have avoided redundant parameters as much as possible. In addition, we have formulated the problem as that of a dynamical system with a constant time delay. These revisions enable us to investigate the mathematical natures of the model from the viewpoint of

the linear stability analysis. Supplementary explanations on the relationship between the variables of the present formulation and those of Barmin et al. (2002) are given in Appendix A.

#### 3. Linear stability analysis for Barmin's model

Dynamic behavior of the nonlinear dynamical system is investigated by analyzing the stability of steady state solutions (i.e., fixed points). In this section, first, we investigate the stability of the fixed points for the dynamical system with time delay (referred to as DSTD) on the basis of the linear stability analysis. Second, we introduce an approximate model for the dynamical system and derive an explicit mathematical expression of the condition for the fixed points to be unstable. Finally, we compare the results of these stability analyses with those of numerical calculations. Because we are concerned only with dimensionless variables in this and next sections, we omit primes from the dimensionless variables in these sections.

3.1. Stability analysis for dynamical system with time delay (DSTD)

In Barmin's model, the fixed points  $(Q_f, P_f)$  are determined by the condition that dP/dt=0 and dQ/dt=0 in Eqs. (10)–(12) and that the position of the viscosity increase is fixed at  $x_{*f}=Q_{in}t_*$  in Eqs. (7)–(9) as

$$\begin{cases} (Q_{\rm f}, P_{\rm f}) = (Q_{\rm in}, \{\hat{\mu} - (\hat{\mu} - 1)Q_{\rm in}t_*\}Q_{\rm in}), & x_{*{\rm f}} < 1\\ (Q_{\rm f}, P_{\rm f}) = (Q_{\rm in}, Q_{\rm in}), & x_{*{\rm f}} > 1. \end{cases}$$
(13)

These relationships are expressed as a parabolic function  $(x_{*f} < 1)$ and a liner function  $(x_{*f} > 1)$  in Q-P phase plane. We refer to these relationships collectively as a steady P-Q curve hereafter. Fig. 2 shows the steady P-Q curve in  $Qt_*-Pt_*$  phase plane



Fig. 2. The fixed points in  $Qt_* - Pt_*$  phase plane. The fixed points for  $x_{*f} < 1$  is expressed by a parabolic function  $(Pt_* = \{\hat{\mu} - (\hat{\mu} - 1)Qt_*\} Qt_*)$ , while the fixed points for  $x_{*f} > 1$  is expressed by a linear function  $(Pt_* = Qt_*)$ . Note that the shape of the parabolic curve (e.g., the values of  $Qt_*$  at A, B and D) depends on  $\hat{\mu}$ ; the parabolic curve for  $\hat{\mu} = 10$  is illustrated in this diagram. Here we rotated the axes of the diagram showing the same relationship in Barmin et al. (2002), because we discuss the stability of the fixed points in terms of  $dP_{f'}dQ_f$  and  $Q_{in}t_*$ .

instead of Q-P phase plane so that the relationships are dependent on only  $\hat{\mu}$ . The parabolic relationship represents the steady magma flow where the effective viscosity decreases as the magma flow rate increases. On the other hand, the linear function represents Poiseuille flow with a constant effective viscosity of  $\bar{\mu} = 1$ . The intersection of the parabola and the liner function is  $(Qt_*, Pt_*) = (1, 1)$  and it is independent of  $\hat{\mu}$  (point C in Fig. 2).

When  $x_{*f} > 1$ , because the influence of time delay disappears and because P(t) = Q(t) in Eq. (7), the problem of the stability of the fixed point is reduced to a problem of a simple ordinary differential equation. Integrating Eq. (12), we obtain the solution as

$$P(t) = Q(t) = Q_0 \exp(-t) + Q_{\rm in},$$
(14)

where  $Q_0$  is the arbitrary constant determined by the initial value of the system. Because  $P=Q=Q_{in}$  for  $t \to \infty$  in Eq. (14), we can conclude that the fixed points for  $x_{*f} > 1$  are stable.

When  $x_{*f} < 1$ , the stability of the fixed point is evaluated by a linear stability analysis for the DSTD. Substituting  $P(t)=P_{\rm f}+\delta P(t)$ ,  $Q(t)=Q_{\rm f}+\delta Q(t)$  and  $Q(t-t_*)=Q_{\rm f}+\delta Q(t-t_*)$  into Eqs. (10) and (11) where  $\delta P$  and  $\delta Q$  are small deviations from the fixed point, and linearizing the expansion in powers of  $\delta P(t)$ ,  $\delta Q(t)$  and  $\delta Q(t-t_*)$ , we obtain

$$\frac{d\delta P(t)}{dt} = -\delta Q(t),\tag{15}$$

and

$$\frac{d\delta Q(t)}{dt} = \frac{Q_{\rm f}}{P_{\rm f}} [\{-1 + (\hat{\mu} - 1)Q_{\rm f}\}\delta Q(t) - (\hat{\mu} - 1)Q_{\rm f}\delta Q(t - t_*)],$$
(16)

respectively. Because the set of Eqs. (10) and (11) is essentially a dynamical system with one variable Q, Eq. (16) dose not include  $\delta P$ ; therefore, the stability of the fixed points is determined only by the behavior of  $\delta Q$ . Assuming solutions of the form  $\delta Q(t) = \delta Q_0 \exp(\lambda t)$  for Eq. (16) (and hence  $\delta P(t) = -\delta Q_0 \exp(\lambda t)/\lambda$  for Eq. (15)) yields

$$\delta Q(t - t_*) = \delta Q(t) \exp(-\lambda t_*), \tag{17}$$

where  $\lambda$  is the eigenvalue of the fixed point (e.g., Farmer, 1982). Substituting these relationships into Eq. (16), we obtain the characteristic equation as

$$\lambda + A + B \exp(-\lambda t_*) = 0, \tag{18}$$

where

$$\begin{cases}
A = \frac{1}{\hat{\mu} - (\hat{\mu} - 1)Q_{\text{in}}t_*} \{1 - (\hat{\mu} - 1)Q_{\text{in}}\} \\
B = \frac{1}{\hat{\mu} - (\hat{\mu} - 1)Q_{\text{in}}t_*} \{(\hat{\mu} - 1)Q_{\text{in}}\}.
\end{cases}$$
(19)

Eq. (18) is transcendental and has infinite complex roots of  $\lambda = \sigma + i\omega$ . Separating Eq. (18) into the real and imaginary parts yields

$$\begin{cases} \sigma + A = -B \exp(-\sigma t_*) \cos(\omega t_*) \\ \omega = B \exp(-\sigma t_*) \sin(\omega t_*) \end{cases}$$
(20)

When the dimensionless parameters of the model (i.e.,  $\hat{\mu}$ ,  $Q_{in}$  and  $t_*$ ) are given,  $\sigma$  and  $\omega$  can be calculated from Eq. (20) by a suitable numerical technique (e.g., the Newton–Raphson iteration method). For the numerical convenience, we rewrite Eq. (20) as

$$\begin{cases} \sigma = -A - \frac{\omega}{\tan(\omega t_*)} \\ \omega = \left\{ B^2 \exp(-2\sigma t_*) - (\sigma + A)^2 \right\}^{1/2}. \end{cases}$$
(21)

Generally the *n*-th largest  $\sigma$  is found in the range of (n-1) $\pi < \omega t_* < n\pi$  in Eqs. (20) or (21). When the first largest real part of  $\lambda$  (i.e., the largest  $\sigma$ ) is positive, the fixed point is concluded to be unstable.

Fig. 3 shows the first largest  $\sigma$  and the second largest  $\sigma$  as a function of time delay  $t_*$  for  $x_{*f} (=Q_{in}t_*)=0.8$  and  $\hat{\mu}=10$ . When  $t_* < t_{*R}$ , no complex root of  $\lambda$  (i.e., real  $\omega$ ) is found in the range of  $0 < \omega t_* < \pi$  in Eqs. (20) or (21); Eq. (18) has two positive real roots of  $\lambda$  in this range. When  $t_* > t_{*R}$ , Eq. (18) has a complex root with the first largest  $\sigma$  in the range of  $0 < \omega t_* < \pi$ . The first largest  $\sigma$  is positive real part for  $t_* < t_{*C}$  and negative for  $t_* > t_{*C}$ . Because  $\lambda$  has a positive real part for  $t_* < t_{*C}$ , the condition of  $t_* = t_{*C}$  ( $t_{*C} = 11.14$  for  $x_{*f} = 0.8$  and  $\hat{\mu} = 10$ ) is referred to as a bifurcation condition.

Fig. 4 shows  $t_{*C}$  as a function of  $x_{*f}$  for  $\hat{\mu}$  varying from 3 to  $10^3$ . For  $\hat{\mu}/\{2(\hat{\mu}-1)\} < x_{*f} < 1$  (i.e., the range between A and C in Fig. 2),  $t_{*C}$  is 0 at  $x_{*f} = \hat{\mu}/\{2(\hat{\mu}-1)\}$  and increase with  $x_{*f}$ . In this range the relationships between the eigenvalues and the time delay are qualitatively same as those of Fig. 3 for all the values of  $\hat{\mu}$ . On the other hand, for  $x_{*f} < \hat{\mu}/\{2(\hat{\mu}-1)\}$ , there is no  $t_*$  for  $\sigma$  to be positive. These results suggest that the fixed point is



Fig. 3. Real parts of eigenvalues as a function of time delay  $t_*$  for  $\hat{\mu} = 10$  and  $x_{*f} = 0.8$ . The first largest and the second largest real parts are shown. When  $t_* < t_{*C} = 11.14$ , the first largest real parts of eigenvalues are positive, and so the fixed point is unstable.

M. Nakanishi, T. Koyaguchi / Journal of Volcanology and Geothermal Research xx (2008) xxx-xxx



Fig. 4. The time delay at the bifurcation condition  $t_{*C}$  as a function of  $x_{*f}$  for different  $\hat{\mu}$ . (a):  $\hat{\mu} = 3$ . (b):  $\hat{\mu} = 10$ . (c):  $\hat{\mu} = 100$ . (d):  $\hat{\mu} = 1000$ . The position of  $x_{*f} = \hat{\mu} / \{2 (\hat{\mu} - 1)\}$  (the point A in Fig. 2) is shown by a thin solid vertical line. The time delay at the bifurcation condition for the approximate model  $t_{*C}^{app}$  (dashed curves) will be discussed in a later subsection.

unstable when  $t_* < t_{*C}$  in the range of  $\hat{\mu}/\{2(\hat{\mu}-1)\} < x_{*f} < 1$ , whereas it is stable when  $x_{*f} < \hat{\mu}/\{2(\hat{\mu}-1)\}$  or  $x_{*f} > 1$ .

In Fig. 4, the values of  $t_{*C}$  are approximated by  $2\hat{\mu}$  at  $x_{*f}=1$  for large values of  $\hat{\mu}$ . This feature reflects the fact that the first largest  $\sigma$  is found in the range of  $0 < \omega t_* < \pi$  in Eqs. (20) or (21), and that  $\omega t_{*C}$  approaches  $\pi$  as  $x_{*f} \rightarrow 1$  (Fig. 5); the relationship between  $x_{*f}$  and  $\omega t_{*C}$  converges a single curve connecting (0.5, 0) and (1,  $\pi$ ) as  $\hat{\mu}$  increases. From Eqs. (19), (20) and (21) and the definition of  $x_{*f}$  ( $\equiv Q_{in}t_*$ ), we obtain  $t_{*C} \rightarrow 2(\hat{\mu} - 1)$  as  $x_{*f} \rightarrow 1$  and  $\omega t_{*C} \rightarrow \pi$ . We will use this nature for the evaluation of the approximate model in a later subsection.

The results of Figs. 3 and 4 indicate that the bifurcation condition is expressed by a critical value of  $t_*$  for a given  $x_{*f}$ . In other words, the bifurcation condition can be expressed by a critical value of  $x_{*f}$  for a given  $t_*$ . As will be discussed in a later section, the value of  $t_*$  is basically determined by the material properties (e.g., viscosity and kinetics of crystallization) and the geophysical conditions (e.g., chamber depth and conduit radius), while  $x_{*f}$  (= $Q_{in}t_*$ ) for a given  $t_*$  depends on eruption condition (i.e.,  $Q_{in}$ ). Therefore, in the following section, we attempt to obtain the range of  $Q_{in}$  (or the range of  $x_{*f}$  in Fig. 2) where the fixed points are unstable for a given  $t_*$ .

#### 3.2. Approximate model

Although the bifurcation condition of the DSTD is accurate, the value of  $t_{*C}$  can be determined only by solving  $\sigma=0$  in Eqs. (20) or (21) numerically. In order to obtain an explicit form of the range of  $x_{*f}$  or  $Q_{in}$  for the unstable fixed point for a given  $t_*$ , we develop an approximate model.

In the DSTD, the effect of time delay is taken into account in Eq. (9) where the position of the viscosity increase  $x_*(t)$  is determined by the integration of Q(t) from  $t-t_*$  to t. In the approximate model, using the trapezoid formula (see Appendix B



Fig. 5. The value of  $\omega t_{*C}$  in Eqs. (20) and (21) as a function of  $x_{*f}$  for  $\hat{\mu} = 3, 10, 100$  and 1000. For all the curves  $\omega t_{*C} \rightarrow 0$  as  $x_{*f} \rightarrow \hat{\mu} / \{2(\hat{\mu} - 1)\}$ .

for the evaluation of this approximation), we approximate Eq. (9) by

$$x_*(t) \sim \frac{Q(t) + Q(t - t_*)}{2} t_*.$$
 (22)

Substituting this equation into Eqs. (7) and (8), we obtain

$$Q(t-t_*) \sim -Q(t) + \frac{2}{(\hat{\mu}-1)t_*} \left\{ \hat{\mu} - \frac{P(t)}{Q(t)} \right\}.$$
 (23)

From this approximation, Eq. (11) is reduced to

$$\frac{dQ}{dt} = \frac{Q}{P} \left[ Q_{\rm in} - Q + \frac{2}{t_*} \left\{ P - F(Q) \right\} \right],\tag{24}$$

where

$$F(Q) = \{\hat{\mu} - (\hat{\mu} - 1)Qt_*\}Q.$$
(25)

Because dQ/dt=0 and  $Q=Q_{in}$  at the steady state, the curve of P=F(Q) is identical to the steady P-Q curve. Using Eq. (24) instead of Eq. (11), the DSTD is reduced to a simple two-dimensional (2-D) dynamical system of Q and P where  $t_*$  is given as a parameter.

On the basis of a linear stability analysis for the 2-D dynamical system, we can obtain a bifurcation condition based on the approximate model. Linearized equations for the approximate model are

$$\frac{d}{dt} \begin{pmatrix} \delta P \\ \delta Q \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ \frac{2Q_{\rm in}}{P_{\rm f}t_*} & \frac{Q_{\rm in}}{P_{\rm f}} \begin{pmatrix} -1 - \frac{2}{t_*} \frac{dF(Q_{\rm in})}{dQ} \end{pmatrix} \end{pmatrix} \begin{pmatrix} \delta P \\ \delta Q \end{pmatrix} \equiv M_{\rm L} \begin{pmatrix} \delta P \\ \delta Q \end{pmatrix}.$$
(26)

The characteristic equation is

$$\lambda^2 - J_{\rm T}\lambda + J_{\rm D} = 0, \qquad (27)$$

where  $J_{\rm T}$  and  $J_{\rm D}$  are the trace and the determinant of  $M_{\rm L}$ , respectively. Because  $Q_{\rm in}$ ,  $P_{\rm f}$  and  $t_*$  are positive, and hence  $J_{\rm D} > 0$ , the stability of the fixed point is dependent on only sign of  $J_{\rm T}$ . When  $J_{\rm T} > 0$ , the fixed points are unstable because the real part of the



Fig. 6. The range of the unstable fixed points on the steady P-Q curve. The fixed points within a shaded area is unstable. The shaded area is determined by the linear stability analysis for the approximate model (see Eqs. (28) and (30)). The parabola for  $\hat{\mu} = 10$  is used for convenience sake in this diagram.



Fig. 7. The regions of the unstable fixed points for the two models in the  $Q_{in}-1/t_*$  space. The region of the unstable fixed points for the DSTD is shown by a shaded area. The region of the unstable fixed points for the approximate model is expressed by an area between two lines of  $x_{*f}=Q_{in}t_*=1$  (dotted line) and  $Q_{in}=Q_{in}^{app}$  (solid line). Although this diagram illustrates the results for the particular case of  $\hat{\mu} = 10$ , the dependence on  $\hat{\mu}$  can be calculated from the relationships based on the approximate model.

eigenvalue is positive. Consequently, the condition for the unstable fixed point is obtained as

$$\frac{dF(Q_{\rm in})}{dQ} < -\frac{t_*}{2},\tag{28}$$

where the left-hand side represents the slope of the steady P-Q curve. The condition indicates that the fixed points become unstable when the negative slope of the steady P-Q curve exceeds a critical value  $t_*/2$  (Fig. 6).

For the approximate model the value of  $t_*$  at the bifurcation condition for given  $\hat{\mu}$  and  $x_{*f}$  is calculated from Eqs. (25) and (28) as,

$$t_{*C}^{\text{app}} = 2[2(\hat{\mu} - 1)x_{*f} - \hat{\mu}].$$
<sup>(29)</sup>

In Fig. 4 both  $t_{*C}$  and  $t_{*C}^{app}$  are 0 at  $x_{*f} = \hat{\mu} / \{2\hat{\mu} - 1\}$  and approach  $2\hat{\mu}$  at  $x_{*f} = 1$  for large  $\hat{\mu}$ ; from Eq. (29)  $t_{*C}^{app} \rightarrow 2(\hat{\mu} - 2)$  as  $x_{*f} \rightarrow 1$ , while  $t_{*C} \rightarrow 2(\hat{\mu} - 1)$  as  $x_{*f} \rightarrow 1$ . Because of this nature,  $t_{*C}^{app}$  fairly agrees with  $t_{*C}$  throughout  $\hat{\mu} / \{2(\hat{\mu} - 1)\} < x_{*f} < 1$  for a wide range of  $\hat{\mu}$  from 3 to 10<sup>3</sup> (see Fig. 4).

As was mentioned above, we are interested in the range of  $x_{*f}$  or  $Q_{in}$  for the unstable fixed point for a given  $t_*$ . Eq. (28) indicates that for  $t_* \sim 0$  the range of the unstable fixed points coincides with that of the negative slope of the steady P-Q curve in Fig. 6, whereas as  $t_*$  increases, the range of the unstable fixed points becomes narrow in this diagram. From Eq. (29) and the fact that the fixed point is stable when  $x_{*f} > 1$ , the range of  $x_{*f}$  for the fixed points to be unstable is obtained as

$$1 > x_{*f} > \frac{\hat{\mu}}{2(\hat{\mu} - 1)} + \frac{t_*}{4(\hat{\mu} - 1)} \equiv x_{*fC}^{app}, \tag{30}$$

and that of  $Q_{in}$  for the unstable fixed points as

$$\frac{1}{t_*} > Q_{\rm in} > \frac{\hat{\mu}}{2(\hat{\mu} - 1)t_*} + \frac{1}{4(\hat{\mu} - 1)} \equiv Q_{\rm inC}^{\rm app}.$$
(31)

We call the condition of  $Q_{in} = Q_{inC}^{app}$  (and hence  $x_{*f} = x_{*iC}^{app}$  and  $t_* = t_{*C}^{app}$ ) the approximate bifurcation condition.

Fig. 7 shows the results of the linear stability analyses for the approximate model and the DSTD in the  $Q_{in}-1/t_*$  space. This parameter space has an advantage that  $x_{*f}$  is expressed by a slope of a straight line from the origin. The region of the unstable fixed points for the approximate model (i.e., Eq. (31)) is expressed by the area between the lines of  $x_{*f}=Q_{in}t_*=1$  and  $Q_{in}=Q_{in}^{app}$  in this diagram. On the other hand, the region of the unstable fixed points for the DSTD is expressed by a shaded area in Fig. 7. The two regions of the unstable fixed points for the DSTD is expressed by a shaded area in Fig. 7 shows the results for the particular case of  $\hat{\mu} = 10$ , the good agreement between the two models in Fig. 4 implies that the relationships based on the approximate model in this diagram are applicable for a wide range of  $\hat{\mu}$ . Thus, the dependence of the stability of the fixed points on all

the three governing dimensionless parameters (i.e.,  $t_*$ ,  $Q_{\rm in}$  and  $\hat{\mu}$ ) are shown in this diagram.

Fig. 7 indicates that there are minimum  $Q_{in}$  and minimum  $1/t_*$  (maximum  $t_*$ ) for the unstable fixed point to exist. They are calculated from the intersection of the two lines of  $Q_{in} = Q_{inC}^{app}$  and  $Q_{in}t_*=1$  as

$$Q_{\rm in} > \frac{1}{2(\hat{\mu} - 2)} \text{ and } \frac{1}{t_*} > \frac{1}{2(\hat{\mu} - 2)}.$$
 (32)

These conditions provide necessary conditions for the fixed points to be unstable.

#### 3.3. Comparison with numerical calculation

So far we determined the bifurcation conditions of the fixed point on the basis of the linear stability analyses for the DSTD



Fig. 8. Numerical results of trajectories in the  $Qt_* - Pt_*$  phase plane and time evolution of  $Pt_*$  for  $\hat{\mu} = 10$ ,  $x_{*t} = 0.8$  with various  $t_*$ . (a), (d): trajectory and time evolution for  $t_* = 1.0$ . (b), (e): trajectory and time evolution for  $t_* = 11$ . (c), (f): trajectory and time evolution for  $t_* = 14$ . Dotted curves: trajectories. Stars: initial values. Thick solid curves in (a) and (b): limit cycles. Thin solid curves in (a), (b) and (c): the steady P - Q curve. Note that the shape of the steady P - Q curve does not depend on  $t_*$ . In (a), the points A, B, C and D in Fig. 2 are plotted on the steady P - Q curve.

and the approximate model. Here we confirm these results and investigate the evolution of the unstable fixed points by calculating the basic equations (Eqs. (10)-(12)) numerically.

Fig. 8 shows numerical results of the evolution of  $(Qt_*, Pt_*)$ in the dynamical system with a fixed point  $(Q_{in}t_*, P_f t_*)=(0.8, 2.24)$  (i.e.,  $x_{*f}=0.8$ ) and  $\hat{\mu}=10$  (c.f., Barmin et al., 2002). For  $\hat{\mu}=10$  and  $x_{*f}=0.8$ , the linear stability analysis for the DSTD suggests that the bifurcation condition is given by  $t_{*C}=11.14$  (see Fig. 3). When  $t_* < t_{*C}$ , the trajectories which start from the vicinity of the fixed point asymptotically converge toward closed curves (Fig. 8a and b). As a result,  $Pt_*$  (and hence,  $Qt_*$ ) shows periodic behavior (Fig. 8d and e). In general, the closed curves in Fig. 8a and b are referred to as limit cycles. When  $t_* > t_{*C}$ , the trajectories spiral to the fixed point (Fig. 8c) and  $Pt_*$  approaches the fixed point with a damped oscillation (Fig. 8f). These numerical results are consistent with the result of the linear stability analysis.

#### 4. Characteristics of limit cycles

The results of Fig. 8 indicate that the unstable fixed points result in the periodic behavior of the dynamical system (i.e., limit cycle). Trajectories of limit cycles are classified into two types: Types A and B. Type A limit cycle has the trajectory to move along the steady P-Q curve in the  $Qt_*-Pt_*$  phase plane as shown in Fig. 8a. The values of P and Q increase slowly along DA, jump from A to B, decrease rapidly along BC, and jump from C to D. In this type, the cyclic pattern of P is characterized by a series of peaks with the slow increase and the rapid decrease (Fig. 8d). On the other hand, Type B limit cycle has the ellipse trajectory in the neighborhood of the fixed point (Fig. 8b) and the sinusoidal pattern (Fig. 8e). Because the general features of the limit cycles have already numerically investigated by Barmin et al. (2002), we focus on the problems which are related to the present linear stability analysis here.



Fig. 9. Dimensionless periods of limit cycles for  $\hat{\mu} = 10$  in the  $Q_{in} - 1/t_*$  space. The limit cycles exist within the region of the unstable fixed point in this diagram. For the regions of the unstable fixed and stable fixed points see Fig. 7. The dotted lines indicate the sections of constant  $x_{*f}$  (0.9 and 0.56), which will be described in Fig. 10. Three vertical dashed lines indicate the increase in the parameter  $Q_{in}$  for three different  $1/t_*$  (0.07, 0.2 and 1.6), which will be used in the bifurcation analyses of Fig. 12.



Fig. 10. Dimensionless periods of limit cycles *T* as a function of  $1/t_*$  for  $\hat{\mu} = 10$  with constant  $x_{*f}$ . (a):  $x_{*f} = 0.9$ . (b):  $x_{*f} = 0.56$ . Dashed-and-dotted lines: the periods of Type A limit cycles ( $T_A$  calculated from Eq. (33)). Closed circles: the periods at  $t_{*C}$  for the DSTD ( $T_B$  calculated from Eqs. (20) or (21)). Open circles: the periods at  $t_{*C}^{app}$  for the approximate model ( $T_B^{app}$  calculated from Eq. (34)). Note that  $T_A$  is independent of  $1/t_*$  in Eq. (33).

#### 4.1. Periods of limit cycles

First, we describe how the dimensionless period of the limit cycle (referred to as T) varies within the region of the unstable fixed point in the  $Q_{in}-1/t_*$  space (see Fig. 7). Fig. 9 shows the variations in T for  $\hat{\mu} = 10$  in the  $Q_{in}-1/t_*$  space. The dimensionless period T decreases with  $1/t_*$  for a given  $x_{*f}$  (see Fig. 10 for the cases of  $x_{*f}=0.9$  and 0.56). Dependence of T on the dimensionless parameters such as  $t_*$  and  $x_{*f}$  is explained by two analytical periods for Type A and Type B limit cycles.

The period of Type A limit cycle (referred to as  $T_A$ ) has been estimated by Barmin et al. (2002), and using our notations, it is rewritten as

$$T_{\rm A} = \frac{\hat{\mu}}{Q_{\rm A}t_*} \left[ Q_{\rm A}t_* - Q_{\rm D}t_* + (x_{*\rm f} - Q_{\rm A}t_*)\ln\left(\frac{x_{*\rm f} - Q_{\rm A}t_*}{x_{*\rm f} - Q_{\rm D}t_*}\right) \right] \\ + \ln\left(\frac{Q_{\rm B}t_* - x_{*\rm f}}{Q_{\rm C}t_* - x_{*\rm f}}\right), \tag{33}$$

where  $Q_A t_*$ ,  $Q_B t_*$ ,  $Q_C t_*$  and  $Q_D t_*$  represent the position of A, B, C and D in the horizontal axis of Fig. 2. The value of  $T_A$  is

estimated from the timescale for *P* and *Q* to move along the steady P-Q curve from D to A and from B to C, and in practice it is calculated from the integration of Eqs. (10) and (13). Because the position of A, B, C and D (i.e.,  $Q_A t_*, Q_B t_*, Q_C t_*$  and  $Q_D t_*$ ) is determined only by  $\hat{\mu}$ , Eq. (33) implies that  $T_A$  is functions of  $\hat{\mu}$  and  $x_{*f}$ ; it is roughly proportional to the value of  $\hat{\mu}$ , although it slightly varies with  $x_{*f}$  as will be shown later.

Although the transition between Type A and Type B limit cycles is gradual, the period of the extreme case of Type B limit cycles may be estimated from the period at the bifurcation condition on the basis of the linear stability analysis (referred to as  $T_{\rm B}$ ). The period at the bifurcation condition for the DSTD is numerically obtained from Eqs. (20) or (21) as  $T_{\rm B}=2\pi/\omega$ . The period at the bifurcation condition for the approximate model is calculated from the imaginary part of the eigenvalue, Im( $\lambda$ ), in Eq. (27). Because Im( $\lambda$ ) =  $\sqrt{J_{\rm D}}$  under the bifurcation condition and because  $J_{\rm D}=2Q_{\rm in}/P_{\rm f} t_*$  in Eq. (26), it is expressed as

$$T_{\rm B}^{\rm app} = \frac{2\pi}{{\rm Im}(\lambda)} = 2\pi \sqrt{\overline{\mu_{\rm f}} \frac{t_{*\rm C}^{\rm app}}{2}},\tag{34}$$

where  $\bar{\mu}_{\rm f} = \hat{\mu} - (\hat{\mu} - 1)x_{*\rm f}$  is the average viscosity at the fixed point and  $t_{*\rm C}^{\rm app}$  is the time delay at the bifurcation condition for the approximate model (see Eq. (29)).

Fig 10 shows the variations of T along  $x_{*f}=0.9$  and  $x_{*f}=0.56$ in Fig. 9. In this diagram  $T_A$ ,  $T_B$  and  $T_B^{app}$  are also plotted. The value of  $T_A$  is constant regardless of  $1/t_*$  for given  $x_{*f}$ . For  $x_{*f}=0.9$ ,  $T_{\rm B}$  and  $T_{\rm B}^{\rm app}$  are greater than  $T_{\rm A}$ . In this case, the period of limit cycle T coincides with  $T_{\rm B}$  at the bifurcation condition and it decreases with increasing  $1/t_*$  asymptotically approaching a constant value of  $T_A$  (Fig. 10a). For  $x_{*f}$ =0.56, on the other hand,  $T_{\rm B}$  (and  $T_{\rm B}^{\rm app}$ ) is smaller than  $T_{\rm A}$  at the bifurcation condition. In this case, the value of T is approximated by  $T_A$  at all the values of  $1/t_*$  including the vicinity of the bifurcation condition (Fig. 10b). These results indicate that the behavior of T around the bifurcation condition depends on whether  $T_A > T_B$  or  $T_A < T_B$ . Although Figs. 9 and 10 illustrate specific results for  $\hat{\mu} = 10$ , this tendency that the behavior of T depends on the relative magnitude of  $T_A$  and  $T_B$  is commonly observed in the results of  $3 < \hat{\mu} < 10^3$ .

Fig. 11 shows  $T_A$ ,  $T_B$  and  $T_B^{app}$  as a function of  $x_{*f}$  for  $\hat{\mu}$  varying from 3 to 10<sup>3</sup>. For a wide range of  $\hat{\mu}$  (particularly for  $\hat{\mu} > 10$ ),  $T_A$  is approximated by  $\hat{\mu}$  and it depends only weakly on  $x_{*f}$  (see also Eq. (33)). On the other hand,  $T_B$  is 0 at  $x_{*f} = \hat{\mu} / \{2(\hat{\mu} - 1)\}$  and increases with increasing  $x_{*f}$ ;  $T_B$  approaches  $\sim 4\hat{\mu}$  as  $x_{*f} \rightarrow 1$  for  $\hat{\mu} > 10$ . From these features of  $T_A$  and  $T_B$ , it is concluded that  $T_A$  is greater than  $T_B$  around  $x_{*f} \sim \hat{\mu} / \{2(\hat{\mu} - 1)\}$ , while  $T_B > T_A$  for the rest of  $x_{*f}$ . In this diagram, we observe



Fig. 11. Dimensionless periods of limit cycles as a function of  $x_{*f}$  for different  $\hat{\mu}$ . (a):  $\hat{\mu} = 3$ . (b):  $\hat{\mu} = 10$ . (c):  $\hat{\mu} = 100$ . (d):  $\hat{\mu} = 1000$ . Dashed-and-dotted curve:  $T_A$  calculated from Eq. (33). Solid curve: the period at the bifurcation condition for the DSTD ( $T_B$ ), which is numerically obtained from Eqs. (20) or (21). Dashed curve: the period at the bifurcation condition for the approximate model ( $T_B^{app}$ ) calculated from Eq. (34). Dotted curve: the period at the bifurcation condition for the hybrid model see Appendix B. The position of  $x_{*f} = \hat{\mu} / \{2(\hat{\mu} - 1)\}$  (the point A in Fig. 2) is shown by a thin solid vertical line.

that  $T_{\rm B}^{\rm app}$  deviates from  $T_{\rm B}$  for large  $x_{\rm sf}$ . The problems related to the difference between  $T_{\rm B}$  and  $T_{\rm B}^{\rm app}$  and the behavior of  $T_{\rm B}$  at  $x_{\rm sf} \sim 1$  are discussed in Appendix B.

The results of Figs. 9, 10 and 11 are summarized as follows. For small  $x_{*f}$  (around the peak of the parabola in Fig. 2), the limit cycle is Type A with  $T \sim T_A$  throughout the region where the fixed point is unstable. When  $x_{*f}$  is substantially greater than that of the peak of the parabola in Fig. 2, the limit cycle is Type B with  $T \sim T_B$  around the bifurcation condition and it approaches to Type A limit cycle with  $T \sim T_A$  with increasing  $1/t_*$ .

#### 4.2. Nature of bifurcation

In Fig. 10 we discussed how the periodic behavior for a given  $x_{*f}$  varies with  $1/t_*$ . Considering that  $1/t_*$  is a fixed parameter for a given magma under a given geological condition, we are more interested in how the periodic behavior for a given  $1/t_*$  varies with  $Q_{in}$  (see the vertical arrows in Fig. 9). The above analyses suggest that the periodic behavior around the bifurcation condition depends on  $x_{*f}$ , and hence  $1/t_*$ . We will briefly describe how the nature of the bifurcation (i.e., the transition from stable steady flow to periodic flow with increasing  $Q_{in}$ ) changes for different  $1/t_*$  below.

Fig. 12 shows the bifurcation diagrams as a function of  $Q_{in}$  for  $\hat{\mu} = 10$  with four different values of  $1/t_*$ . For small  $1/t_*$  (Fig.

12a), as  $Q_{\rm in}$  increases, the transition from stable steady flow to periodic flow occurs gradually. On the other hand, for large  $1/t_*$  (Fig. 12b, c and d), the transition occurs discontinuously as  $Q_{\rm in}$  increases.

The above two types of bifurcations correspond to the supercritical Hopf bifurcation and the subcritical Hopf bifurcation in the general problem of the 2-D dynamical systems. In the supercritical Hopf bifurcation, as the control parameter changes, a stable fixed point changes into a stable limit cycle and an unstable fixed point; the amplitude of the stable limit cycle increases gradually (e.g., see Fig. 12a). In the subcritical Hopf bifurcation, on the other hand, both the stable fixed point and the stable limit cycle coexist under the same control parameter in the vicinity of the bifurcation point (e.g., see Fig. 12b). Which stable solution is chosen depends on the initial condition. In this case, the transition from the stable fixed point to the stable limit cycle coexist under the stable fixed point to the stable limit cycle can occur discontinuously with continuously varying control parameter.

The value of  $1/t_*$  where the transition from subcritical to supercritical bifurcation occurs roughly coincides with that of  $1/t_*$  where the transition from Type A limit cycle to Type B limit cycle occurs. Accordingly, the nature of the bifurcation can be summarized as follows. When  $1/t_*$  is large, the transition from steady flow to periodic flow of Type A limit cycle occurs discontinuously (i.e., the subcritical bifurcation). On the other



Fig. 12. Bifurcation diagram as a function of  $Q_{in}$  for  $\hat{\mu} = 10$  with four different values of time delay  $t_*$ : (a)  $1/t_* = 0.07$ ; (b)  $1/t_* = 0.2$ ; (c)  $1/t_* = 1.6$ ; (d)  $1/t_* = 50$ . Solid curves and dotted curves are stable solutions and unstable solutions, respectively. These curves are obtained by calculating the basic equations (Eqs. (7)–(12)) numerically. There are two types of the bifurcations: supercritical Hopf bifurcation (a) and subcritical Hopf bifurcation (b,c).

hand, when the  $1/t_*$  is small, the transition from steady flow to periodic flow of Type B limit cycle occurs gradually (i.e., the supercritical bifurcation). The value of  $1/t_*$  that separates the above two situations may be crudely estimated from the condition of  $T_A = T_B^{app}$  for Eq. (34) (note that  $T_B^{app} \sim T_B$  at the point of  $T_A = T_B$  in Fig. 11) as

$$\frac{1}{t_{*C}^{app}}\Big|_{T_{A}=T_{B}} = \frac{2\pi^{2}\,\overline{\mu}_{f}}{T_{A}^{2}}.$$
(35)

Using the approximations of  $T_A \sim \hat{\mu}$  (see Fig. 11) and  $\bar{\mu}_f \sim \hat{\mu}/2$ around  $x_{*f} \sim \hat{\mu}/\{2(\hat{\mu}-1)\}$  (see Eq. (8)), we can roughly evaluate the right-hand side of Eq. (35) as

$$\frac{1}{t_{*C}^{app}}\Big|_{T_{A}=T_{B}} \sim \frac{10}{\hat{\mu}}.$$
(36)

Eq. (36) is a crude approximation that may vary by an order of magnitude; our numerical results indicate that the subcritical bifurcation tends to occur at smaller values of  $1/t_*$  than is predicted by this equation. Nevertheless, this approximation would be useful in quasi-quantitatively predicting the nature of bifurcation and the types of limit cycles around the bifurcation condition in the  $Q_{in}-1/t_*$  space when  $1/t_*$  varies over several orders of magnitude.

#### 5. Geophysical implication

So far we have analyzed the problem using dimensionless parameters for mathematical convenience. In order to apply the results of the analyses to geophysical situations, we rewrite those results in the dimensional form using the characteristic scales in Eq. (6). In the foregoing sections, the stabilities of the fixed points are determined by the three dimensionless parameters of  $1/t'_*(=\overline{T}/t_*)$ ,  $Q'_{in}(=Q_{in}/\overline{Q})$  and  $\hat{\mu}(=\mu_2/\mu_1)$ . Among these parameters,  $1/t'_*$  governs the nature of bifurcation from steady flow to periodic flow as well as the types of trajectories and the periods of limit cycles (e.g.,  $T'_A (=T_A/\overline{T})$ and  $T'_B(=T_B/\overline{T})$ ). Considering these results, it is particularly important to understand the physical meaning of  $\overline{T}$ .

In order to understand the physical meaning of  $\overline{T}$ , we discuss the physical meaning of  $\overline{P}$  first. The physical meaning of  $\overline{P}$ comes from the fact that the oscillation of the conduit flow is driven by the chamber overpressure  $(p_{ch} - \rho gL)$  due to elastic deformation;  $\overline{P}$  is the pressure change at the magma chamber when a magma with the volume of conduit  $(\pi r^2 L)$  is added in or effused from the magma chamber with  $V_{ch}$  (see Fig. 1 for the notations of the variables). When  $\overline{P}$  is given, the characteristic velocity of Poiseuille flow can be defined by

$$\overline{U} = \frac{r^2 \,\overline{P}}{8\mu_1 L},\tag{37}$$

which yields the characteristic flow rate  $\overline{Q}$  and the characteristic timescale  $\overline{T}$  as

$$\overline{Q} = \pi r^2 \overline{U} \tag{38}$$

and

$$\overline{T} = \frac{L}{\overline{U}},\tag{39}$$

respectively. Because of this physical meaning of  $T^-$ , the dimensionless parameters are expressed as complex combinations of quantities related to material properties (e.g.,  $\gamma$ ,  $\mu_1$ ) and those related to geophysical conditions (r, L,  $V_{ch}$ ). For example,  $1/t'_*$  is expressed by the product of a dimensionless parameter related to material properties and that related to the size of the system as

$$\frac{1}{t'_*} = \frac{8\mu_1}{\pi\gamma t_*} \times \frac{LV_{\rm ch}}{r^4} \tag{40}$$

Here, we consider that  $t_*$  is primarily determined by material properties such as the kinetics of crystal growth.

When the petrological features of effused magma are known, the values of material properties are constrained; on the other hand, the size of the system which generates the periodic behavior under consideration may vary from one volcano to another. For  $L=10^3-10^4$  m,  $V_{\rm ch}=10^6-10^{11}$  m<sup>3</sup> and  $r=10^1-10^2$  m,  $LV_{\rm ch}/r^4$  varies from  $10^1$  to  $10^{11}$ . Accordingly, even for the same material properties (e.g., the same magma composition), a wide variation of  $1/t'_*$  is expected because of the wide variation of  $LV_{\rm ch}/r^4$ .

Let us consider how the periodic behavior varies with  $LV_{ch}/r^4$ . According to Fig. 9, the dimensional periods of limit cycles depend on  $1/t'_*$ , and hence  $LV_{ch}/r^4$ . Fig. 13 shows the numerical results of the dimensionless period T' normalized by the dimensionless time delay  $t'_*$  in the Q'in  $-1/t'_*$  space. Because both  $t'_*$  and T' are normalized by  $\overline{T}$ , the ratio of  $T'/t'_*$  directly represents the ratio of dimensional period and time delay  $T/t_*$ . For a given dimensional time delay  $t_*$ , the value of  $T/t_*$  is proportional to



Fig. 13. The variation of periods of limit cycles normalized by time delay in the in  $Q_{in}'-1/t'_*$  space for  $\hat{\mu}=10$ . For the regions of the unstable fixed and stable fixed points see Fig. 7. Because both  $t'_*$  and T' are normalized by  $T^-$ ,  $T'/t'_*$  directly represents dimensional  $T/t_*$ .

the dimensional period *T*. The result of Fig. 13 indicates the dimensional period *T* increases as  $LV_{ch}/r^4$  increases.

The analyses in Section 4 suggest that the type of limit cycles and the nature of the bifurcation depend on  $LV_{ch}/r^4$  for a given magma; the transition from steady flow to periodic flow of Type A limit cycle occurs discontinuously for large  $LV_{ch}/r^4$ , while the transition from steady flow to periodic flow of Type B limit cycle occurs gradually for small  $LV_{ch}/r^4$ . The value of  $LV_{ch}/r^4$ that separates the two cases can be crudely estimated from Eqs. (36) and (40) as

$$\frac{LV_{\rm ch}}{r^4}\Big|_{T_{\rm A}=T_{\rm B}} \sim \frac{10\pi\gamma t_*}{8\mu_2}.$$
(41)

The critical value of  $LV_{\rm ch}/r^4$  is inversely proportional to  $\mu_2$  for given  $t_*$  and  $\gamma$  (note that the dependence on  $\mu_1$  is canceled here). For lava dome eruptions with  $\gamma = 10^{10-11}$  Pa (Melnik and Sparks, 2005),  $t_* = 10^{5-6}$  s (e.g., Barmin et al., 2002; Melnik and Sparks, 2005), the right hand side of this equation is evaluated by  $10^{15-17}/\mu_2$  (in Pa s). For larger  $LV_{\rm ch}/r^4$  than the value of Eq. (41), the period of the limit cycle is well approximated by  $T_{\rm A}$ :

$$T_{\rm A} \sim \frac{8\mu_2 L V_{\rm ch}}{\pi\gamma} r^4 \tag{42}$$

Here we use the approximation of  $T'_{\rm A} \sim \mu_2/\mu_1$  (see Fig. 11). The period of the limit cycle is roughly proportional to  $LV_{\rm ch}/r^4$  and  $\mu_2$  for given  $\gamma$  in this range.

#### 6. Concluding remarks

We investigated the condition for the occurrence of periodic behavior of lava dome eruptions on the basis of a model by Barmin et al. (2002). In Barmin's model, the periodic behavior is driven by the temporal and spatial changes of the viscosity, and the viscosity change is described by a series of equations related to the kinetics of crystal growth and the dependence of viscosity on crystal contents. We found that the mathematical features of this model are independent of details of the kinetics of crystal growth or those of the dependence of viscosity on crystal contents; they are determined only by the fact that the viscosity of each fluid particle increases after the fluid particle ascends for a constant time  $t_*$  from the magma chamber. Thus, the condition for the periodic behavior to occur is determined by the linear stability analyses for the dynamical system with time delay (DSTD) and the approximate model. Our analyses allow us to predict that the condition for the periodic behavior to occur from the nonlinearities in the steady state relation between Q and P(i.e., the steady P-Q curve) and the time delay of viscosity increase. The nature of the periodic behavior of lava dome eruptions is largely controlled by a dimensionless parameter of  $LV_{\rm ch}/r^4$ . When  $LV_{\rm ch}/r^4$  is small, transition from steady to periodic flow with the sinusoidal oscillation (Type B limit cycle) occurs gradually. When  $LV_{ch}/r^4$  is large, the periodic behavior is characterized by a series of peaks with the slow increase and the rapid decrease (Type A limit cycle) and the transition may occur discontinuously. We confirmed that the results of these analyses are robust for a wide range of viscosity ratio (i.e., at least for  $3 < \mu_2/\mu_1 < 10^3$ ).

Generally, a magma plumbing system in nature is composed of complex combinations of dykes and conduits and chambers with different levels and dimensions. Observed periodic behavior during lava dome eruptions is much more complex than those discussed here. For example, oscillations of effusion rate and periodic ground deformation with multiple timescales can be observed at the same volcano even during the same eruption (e.g., Yamashina et al., 1999; Voight et al., 1998). Such periodic variations with periods of different timescales may indicate that they are driven by different mechanisms; Costa et al. (2007b) have suggested that the periodic variation with timescale of several weeks is controlled by the elastic properties of the dyke walls, whereas that with timescale of several months to years would be related to magma chamber pressure. We consider that the model discussed here is relevant to the latter situation. The results of the present analyses would be useful in interpreting the relationships between the geological conditions of magma plumbing systems (e.g., the values of  $LV_{\rm ch}/r^4$ ,  $\gamma$  and  $t_*$ ) and the observed oscillation of effusion rate during lava dome eruptions.

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### Appendix A. The relationships between the variables in the present formulation and those in Barmin et al. (2002)

In both the present formulation and Barmin et al. (2002), the basic equations are normalized on the basis of the idea that, when a certain characteristic pressure  $\overline{P}$  is given, the characteristic velocity and time are given by Eqs. (37) and (39), respectively (in Barmin et al. (2002) the symbols of  $u_*$  and  $t_*$  are used for U and T, respectively). The main difference between the two formulations is that in the physical meaning of the characteristic pressure. In Barmin et al. (2002) the characteristic pressure is given by the hydrostatic pressure ( $\rho gL$ ), whereas we use the definition of Eq. (6). The ratio between the two characteristic pressures is additionally introduced as a dimensionless parameter  $\kappa$  in Barmin et al. (2002); however, this parameter is redundant from the mathematical viewpoint (note that the basic equations (i.e., Eqs. (2)–(5)) do not include hydrostatic pressure explicitly). Because of this difference, our formulation includes three dimensionless parameters  $(\hat{\mu}, O'$  in and  $t'_*)$ , whereas four dimensionless parameters are used in Barmin et al. (2002).

The time delay  $t_*$  in our formulation is expressed using the crystal growth model in Barmin et al. (2002) by

$$t_* = \frac{\beta_*^{1/3} - \beta_{\rm ch}^{1/3}}{\{(4/3)\pi n_{\rm ch}\}^{1/3}\chi},\tag{A.1}$$

where  $\beta_{ch}$  is the volume concentration of crystals at the magma chamber,  $\beta_*$  is the critical volume concentration of crystals at which the bulk viscosity increase from  $\mu_1$  to  $\mu_2$ ,  $n_{ch}$  is the number density of crystals, and  $\chi$  is the linear crystal growth rate. Accordingly, the dimensionless parameter  $t'_*$  in our formulation corresponds to  $1/\Psi$  in Barmin et al. (2002), although the two parameters are normalized by the different characteristic timescales because of the above difference in the characteristic pressures.

## **B.** Evaluation of the difference between the DSTD and the approximate model

The approximate model in Section 3.2 is based on the trapezoid formula,

$$x_*(t) = \int_{t-t_*}^t Q(s) ds \sim \frac{Q(t) + Q(t-t_*)}{2} t_* \equiv x_*^{\text{app}}.$$
 (B.1)

Here we briefly evaluate the difference between the DSTD and the approximate model caused by the difference between  $x_*$  and  $x_*^{app}$ .

When Q(t) oscillates with an amplitude  $\Theta_0$  and it has the form of

$$Q(t) = Q_{\rm f} + \Theta_0 \sin\omega t, \tag{B.2}$$

the leading term of the difference between  $x_*$  and  $x_*^{app}$  is proportional to  $(t_*/T)^2$  (or  $t_*^3$ ), namely

$$x_{*}(t) - x_{*}^{\text{app}} = \frac{\pi^{2} \Theta_{0} t_{*} \sin \omega t}{3} \left(\frac{t_{*}}{T}\right)^{2} + O\left(\left(t_{*}/T\right)^{3}\right).$$
(B.3)

The value of  $x_*(t) - x_*^{\text{app}}$  is limited, because the first largest  $\sigma$  is found in the range of  $0 < \omega t_* < \pi$  in Eqs. (20) or (21), and hence  $0 < t_*/T < 1/2$  (see Fig. 5). Therefore, the trapezoid formula is expected to be a good approximation in the range where  $|t_*/T| \ll 1$  (i.e.,  $x_{*f} \sim \mu^{\gamma} \{2(\mu^{-1})\})$ , whereas the difference between the DSTD and the approximate model can be substantial for  $x_{*f} \sim 1$  where  $(t_*/T)^2$  is as large as 1/4.

Fig. 4 shows that the effect of the difference between  $x_*$  and  $x_*^{\text{app}}$  on the bifurcation condition is insignificant; the relationship between  $t_{*\text{C}}$  and  $x_{*\text{f}}$  in the approximate model agrees well with that of the DSTD for wide ranges of  $x_{*\text{f}}$  and  $\hat{\mu}$ . In contrast, the period at the bifurcation point for the approximate model  $(T_{\text{B}}^{\text{app}})$  deviates from that for the DSTD  $(T_{\text{B}})$  around  $x_{*\text{f}} \sim 1$  (Fig. 11). This deviation comes from the fact that  $T_{\text{B}}$  is strongly constrained by the condition that the first largest  $\sigma$  is found only around  $\omega t_* \sim \pi$  (and hence  $T_{\text{B}} \sim 2t_{*\text{C}} \sim 4\mu$ ) for  $x_{*\text{f}} \sim 1$  in the DSTD (see Figs. 4 and 5), whereas such a constraint does not exist in the approximate model. In order to approximate  $T_{\text{B}}$ without losing this constraint, we can calculate the period at the bifurcation point from Eq. (20) with  $\sigma = 0$  and Eq. (29) (referred to as the hybrid model) as

$$T_{\rm B}^{\rm hyb} = \frac{1}{2[2(\hat{\mu}-1)x_{*\rm f}-\hat{\mu}]} \,\,\arccos\,\left(1 - \frac{2[2(\hat{\mu}-1)x_{*\rm f}-\hat{\mu}]}{(\hat{\mu}-1)x_{*\rm f}}\right). \tag{B.4}$$

Fig. 11 shows that  $T_{\rm B}^{\rm hyb}$  agrees better with  $T_{\rm B}$  than  $T_{\rm B}^{\rm app}$ , although we prefer to use the simpler form of  $T_{\rm B}^{\rm app}$  (i.e., Eq. (34)) in the quasi-quantitative estimation of the condition for  $T_{\rm A} \sim T_{\rm B}$  in Eq. (35).

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M. Nakanishi, T. Koyaguchi / Journal of Volcanology and Geothermal Research xx (2008) xxx-xxx

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