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DSM complete synthetic seismograms: SH, spherically symmetric, case

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Abstract. We present a new technique, based on the Direct Solution Method (DSM) [*Geller et al.*, 1990; *Hara et al.*, 1991; *Geller and Ohminato*, 1994], for calculating SH (toroidal) synthetic seismograms for spherically symmetric, isotropic, media. No asymptotic approximations are used and the synthetics, which fully include both body and surface waves, can be computed for a broad range of frequencies. Our algorithm accurately handles sources that require discontinuities in the vertically dependent part of the displacement. We use matrix operators that minimize the numerical error of the solutions. An example synthetic profile is presented for a 600km deep source in the IASP91 model for the period range 4-5000s.

Introduction

The inversion of seismic waveform data for global three-dimensional (3-D) Earth structure has been almost entirely restricted to long period surface wave data. Body wave data allow higher resolution of the fine structure of the earth's interior. However, due to computational limitations, intermediate quantities such as travel times or cross-correlation delay times, rather than the body wave waveforms themselves, have almost always been used as the data in Earth structure studies.

The ability to model the entire seismic waveform as recorded by broadband, high dynamic range, seismographs [*Wielandt and Steim*, 1986] should lead to better models of Earth structure. However, algorithms which are suitable not only for modeling but also for inverting such waveform data are required.

Existing methods for computing synthetic seismograms at teleseismic distances generally produce only a certain portion of the seismic waveform, e.g., free oscillations, surface waves, or body waves. Long period body wave seismograms are frequently computed using the reflectivity method and an earth flattening transformation [e.g., *Müller and Kind*, 1976]. Some efforts are now being made to compute complete synthetic seismograms by modal superposition; however, as noted by *Geller and Ohminato* [1994], modal superposition is ill-suited to the computation of complete synthetics due to the large number of modes that must be summed.

In this paper we present a method for calculating synthetic SH (toroidal) seismograms for a spherically symmetric, isotropic, Earth model. The extension to the P-SV (spheroidal) case for spherically symmetric media will be presented in a companion paper. Our method can be extended to the laterally heterogeneous case in a straightforward manner [*Cummins et al.*, 1992].

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Theory

In this paper, lower case roman subscripts i, j, k, denote cartesian vector components, and subscripts r, θ, ϕ, denote spherical vector components. We use Greek subscripts and superscripts to denote components in the abstract vector space of trial functions. In this paper these Greek indices correspond to a triplet of indices k, ℓ, and m for the index of the radial trial function, the angular order, and the azimuthal order, respectively.

The DSM is a Galerkin weak form realization of the Method of Weighted Residuals (MWR) [*Geller and Ohminato*, 1994]. We use vector trial functions whose complex conjugates serve as the weight functions. The displacement is represented as a linear combination of the trial functions:

$$u_i = \sum_{\alpha=1}^{N} c_\alpha \Phi_i^{(\alpha)}. \qquad (1)$$

As shown by *Geller and Ohminato* [1994], the DSM transforms the weak form of the elastic equation of motion to the following system of linear equations,

$$(\omega^2 \mathbf{T} - \mathbf{H})\mathbf{c} = -\mathbf{g}, \qquad (2)$$

where $\mathbf{T}$ is the mass (kinetic energy) matrix, $\mathbf{H}$ is the stiffness (potential energy) matrix, and $\mathbf{g}$ is the excitation vector.

The vector trial functions for the toroidal case are

$$\boldsymbol{\Phi}^{(k\ell m)} = \frac{W_k(r)}{L}\left(0\,,\,\frac{1}{\sin\theta}\frac{\partial Y_{\ell m}}{\partial\phi}\,,\,-\frac{\partial Y_{\ell m}}{\partial\theta}\right), \qquad (3)$$

where $Y_{\ell m}(\theta,\phi)$ is a (fully normalized) surface spherical harmonic having angular order ℓ and azimuthal order m, k is the index for the radial dependence of the basis functions, and $L = \sqrt{\ell(\ell+1)}$. See *Press et al.* [1986, p.181] for a complete description of the spherical harmonics and associated Legendre polynomials.

We choose linear splines as the radially dependent part of the trial functions. Their explicit form is

$$W_k(r) = \begin{cases} (r - r_{k-1})/(r_k - r_{k-1}) & r_{k-1} < r \le r_k \\ (r_{k+1} - r)/(r_{k+1} - r_k) & r_k \le r < r_{k+1} \\ 0 & \text{otherwise,} \end{cases} \quad (4)$$

where $r_1 < r_2 \cdots < r_N$. The first and second lines of (4) are ignored for $k = 1$ and $k = N$, respectively.

For these trial functions, the toroidal part of (2) is block diagonal; each block corresponds to a pair of indices ℓ and m and is decoupled from all other blocks. (For the 3-D case the various angular and azimuthal orders are coupled, as are toroidal and spheroidal trial functions, resulting in much larger systems of equations; see *Geller and Ohminato* [1994, Appendix A].)

The matrices $\mathbf{T}$ and $\mathbf{H}$ depend only on ℓ, but the excitation vector, $\mathbf{g}$, depends on both ℓ and m. The decoupled system of equations obtained from (2) for each ℓ and m can be written as:

$$\left(\omega^2\mathbf{T} - \mathbf{H}^{(1)} + \mathbf{H}^{(2)} - (L^2 - 1)\mathbf{H}^{(3)}\right)\mathbf{c} = -\mathbf{g}, \quad (5)$$

where $\mathbf{c}$ and $\mathbf{g}$ are now vectors for a particular ℓ and m. The explicit form of the matrix elements is:

$$\begin{aligned} \mathrm{T}_{k'k} &= \int dr \rho r^2 W_{k'} W_k \\ \mathrm{H}^{(1)}_{k'k} &= \int dr\, \mu r^2 \dot{W}_{k'} \dot{W}_k \\ \mathrm{H}^{(2)}_{k'k} &= \int dr\, \mu r(\dot{W}_{k'} W_k + W_{k'} \dot{W}_k) \\ \mathrm{H}^{(3)}_{k'k} &= \int dr\, \mu W_{k'} W_k, \end{aligned} \quad (6)$$

where μ is the rigidity, ρ is the density, and the dot indicates differentiation with respect to r. The integrals are taken from the CMB to the Earth's surface.

The vertically dependent part of the solution (sometimes denoted by y_1^T) is given by

$$W(r) = \sum_k c_k W_k(r). \quad (7)$$

Excitation

For a point moment tensor on the z-axis ($r = r_s$, $\phi = 0$, $\theta \to 0$), the r.h.s. of (5) is zero except for $m = \pm 1$ or $m = \pm 2$, for which, respectively,

$$g_k = \begin{cases} b_1(iM_{r\theta} \pm M_{r\phi}) \left[\dot{W}_k - W_k/r\right]_{r=r_s} \\ b_2(\mp iM_{\theta\theta} \pm iM_{\phi\phi} - 2M_{\theta\phi}) \left[W_k/r\right]_{r=r_s}, \end{cases} \quad (8)$$

where $b_1 = ((2\ell + 1)/(16\pi))^{1/2}$ and $b_2 = ((2\ell + 1)(\ell - 1)(\ell + 2)/(64\pi))^{1/2}$. Satisfactory solutions for the $m = \pm 2$ case are obtained by solving (5) using (8), as the discontinuity of the traction, $\mu(\dot{W} - W/r)$, at $r = r_s$ is a natural boundary condition.

The body force for the $m = \pm 1$ case is kinematically equivalent to requiring $W(r)$ to be discontinuous at the source depth [e.g., *Takeuchi and Saito*, 1972, p.290]:

$$D = W(r_s^+) - W(r_s^-) = b_1(iM_{r\theta} \pm M_{r\phi})/[r_s^2 \mu(r_s)]. \quad (9)$$

But, since (6) (implicitly) requires $W(r)$ to be a continuous function of depth, the convergence for the $m = \pm 1$ case is suboptimal [*Strang and Fix*, 1973].

To obtain more accurate solutions for the $m = \pm 1$ case we eliminate the body force (8) and instead require $W(r)$ to satisfy the essential boundary condition (9). We introduce a new dependent variable, $X(r)$, which is a continuous function of depth:

$$X(r) = W(r) - DH(r - r_s), \quad (10)$$

where $H(r - r_s)$ is a Heaviside step function. We expand $X(r)$ in terms of the trial functions:

$$X(r) = \sum_k c'_k W_k(r). \quad (11)$$

We define a new matrix operator, $\mathbf{A}$:

$$A_{kk'} = \omega^2 T_{kk'} - H^{(1)}_{kk'} + H^{(2)}_{kk'} - (L^2 - 1)H^{(3)}_{kk'}, \quad (12)$$

where the matrices on the r.h.s. of (12) are defined the same way as in (6), except that the integrals are taken only from r_s to the Earth's surface, rather than from the CMB to the Earth's surface. We also define a force vector $\mathbf{g}'$, whose elements are given by

$$g'_k = D \sum_{p=1}^{N} A_{kp}. \quad (13)$$

We find the expansion coefficients for $X(r)$ by solving

$$\left(\omega^2\mathbf{T} - \mathbf{H}^{(1)} + \mathbf{H}^{(2)} - (L^2 - 1)\mathbf{H}^{(3)}\right)\mathbf{c}' = -\mathbf{g}'. \quad (14)$$

Finally, having found $X(r)$, we obtain $W(r)$ from (10).

Modified Operators

For a flat-layered medium, the error due to using linear splines as the trial and weight functions can be minimized by modifying the definition of the matrix elements [*Toda et al.*, 1992]. In this paper we treat the terms which are dominant for large ℓ by analogy to the flat-layered case. We define the "lumped mass," ρ_l, and the "lumped rigidity," μ_l,

$$\rho_l = \sum_{i=1}^{N} m_i \delta(r - r_i) \quad , \quad \mu_l = \sum_{i=1}^{N} s_i \delta(r - r_i), \quad (15)$$

where

$$(m_i, s_i) = \begin{cases} r_1^{-2} \int_{r_1}^{r_1+\Delta_1} (\rho, \mu) r^2 dr & i = 1 \\ r_N^{-2} \int_{r_N-\Delta_N}^{r_N} (\rho, \mu) r^2 dr & i = N \\ r_i^{-2} \int_{r_i-\Delta_i}^{r_i+\Delta_{i+1}} (\rho, \mu) r^2 dr & \text{otherwise} \end{cases} \quad (16)$$

and $\Delta_i = (r_i - r_{i-1})/2$. We define modified operators

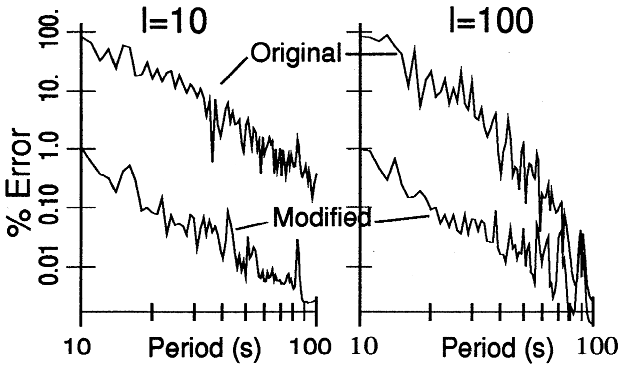

Figure 1. Error for DSM solutions (with $N = 800$) for the IASP91 model.

$$\begin{aligned} \mathrm{T}_{k'k}^{mod} &= \int dr\, \tfrac{1}{2}(\rho + \rho_l) r^2 W_{k'} W_k \\ \mathrm{H}_{k'k}^{(3)mod} &= \int dr\, \tfrac{1}{2}(\mu + \mu_l) W_{k'} W_k . \end{aligned} \qquad (17)$$

We use the modified operators (17) in (5) in place of the original operators defined in (6). $\mathbf{H}^{(1)}$ and $\mathbf{H}^{(2)}$ are not modified. Note that *Marfurt* [1984] and *Korn* [1987] also proposed modified operators similar to those given in (17), but their derivations were based on minimizing numerical dispersion rather than minimizing the error of the numerical solution.

We compare solutions obtained by our method to solutions obtained by numerical integration of the strong form of the equation of motion [*Takeuchi and Saito*, 1972]. An extremely fine grid was used for the latter, which may therefore be regarded as exact. The square of the absolute value of the difference is integrated over radius and the result is divided by the integral of the squared absolute value of the strong form solution to compute the error shown in Figure 1. The moment tensor source has a δ-function time history and corresponds to slip along a horizontal fault in the $\phi = 0$ direction; it thus excites only $m = \pm 1$ waves. We placed the source at a depth of 600km to minimize the excitation of surface waves; the grid spacing was chosen so that the source depth was a node. The use of the modified operators (17) reduces the error by about two orders of magnitude over that for the original operators (6). Note that (14) rather than (5) was used for both the original and modified operators.

Numerical Example

Figure 2 is a record section of ϕ-component (transverse component) synthetics for the spherically symmetric Earth model IASP91 [*Kennett and Engdahl*, 1991], with a density and Q structure which are based on the Earth model PREM [*Dziewonski and Anderson*, 1981]. Physical dispersion is included by treating the IASP91 velocities as referring to a fiducial period of 1s, and using the dispersion relation for a standard linear solid. The source is the same as for Figure 1. The synthetics are computed for a receiver profile in the $\phi = 90°$ direction by solving (5) for all significant values of ℓ and m, summing, and then using the inverse FFT to obtain the solutions in the time domain. A four-pole Butterworth low-pass filter with a corner frequency of 0.125 Hz has been used to avoid a sharp frequency cut-off at the 0.25 Hz Nyquist frequency.

The direct waves as well as those generated by interactions with the free surface and CMB are easily identified by comparison with the arrival times calculated for the IASP91 model using the method of *Sambridge and Kennett* [1990], which are superposed on the waveforms in Figure 2. The travel time curves plotted include the direct wave and the first three free surface reflections

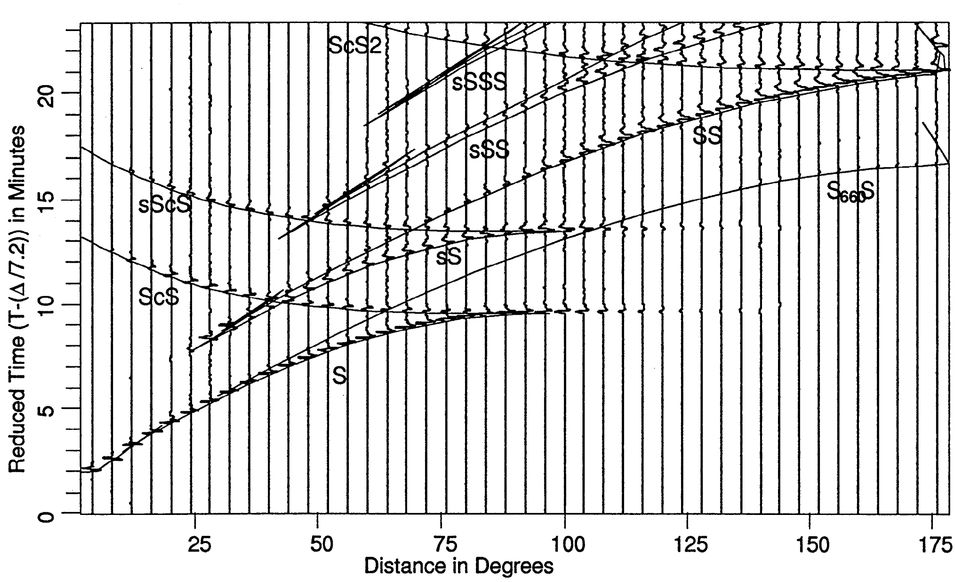

Figure 2. Synthetic transverse component (SH) record section for a 600 km deep source calculated for the IASP91 model. Q structure is that of PREM and the sampling rate is 2s. No instrument response is included, but a low-pass, 4-pole Butterworth filter with a corner at 0.125 Hz is applied.

Table 1. Computational requirements

N	Freq (mHz)	N_f	CPU (s)	Cum (s)	ℓ_{max}
800	25	128	1266	1266	256
1600	50	128	4235	5501	439
2400	75	128	9665	15166	625
3200	100	128	17118	32284	805
4800	150	256	69796	102080	1150
6400	200	256	125542	227622	1484
8000	250	256	196598	424220	1822

Number of layers (N), maximum frequency (Freq), number of frequencies for each band (N_f), computation time (CPU), cumulative computation time (Cum), and angular order at which the calculation was truncated (ℓ_{max}). A Sparc-10 workstation was used.

(i.e., sS-SS, sSS-SSS, sSSS-SSSS) as well as ScS, sScS, and ScS_2. Since considerable energy is radiated downward by the source, core-mantle boundary interactions are quite prominent. The excellent agreement between the synthetics and the travel times, which were calculated completely independently, provides further confirmation of the accuracy of the synthetics.

The calculated waveforms show that, as expected, polarity is preserved on interaction with the free surface (e.g., S and ScS have the same polarity). sS has polarity opposite to that of S because the upgoing "s" takes off from a different lobe of the radiation pattern. As noted by *Choy and Richards* [1975], the SS waveform should be the Hilbert transform of S, and this is evident in Figure 2. Finally, note that diffracted S-arrivals are clearly present in the shadow zone beyond 100°.

$S_{660}S$ is basically the same as SS, but the intermediate reflection is from the underside of the 660 km discontinuity rather than the Earth's surface. Travel time data from such upper mantle reflections provide important data on the fine structure of the mantle [*Shearer*, 1993]. Note that $S_{660}S$ is clearly visible in Figure 2.

The computation time required to produce the synthetics of Figure 2 is given in Table 1. The calculation, which is not yet fully optimized, was carried to a Nyquist frequency of 0.25 Hz. The CPU time required for a Nyquist frequency of 50 mHz (a Nyquist period of 20s) was about 1.5 hr on a Sparc-10 workstation.

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